Asymptotic methods in performance modelling of finite-source retrial queues with collisions and their applications in smart city networks

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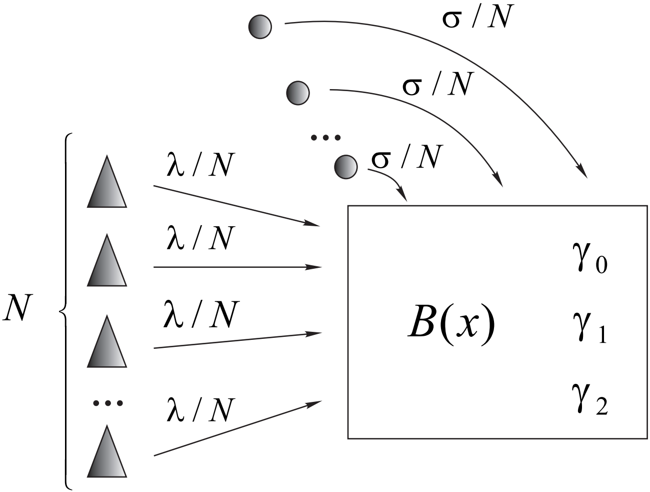
Outline

Finite source retrial queueing system with collisions



Performance measures

- *Distribution of number of requests in the system, including in service and in orbit*
- *Distribution of number of retrials*
- *Distribution of the response/waiting time of a customer*

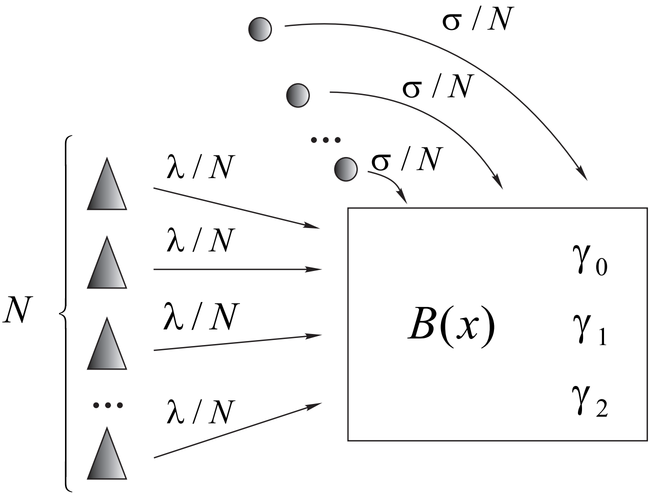
Tool supported and algorithmic approaches

- *MOSEL (Modeling, Specification and Evaluation Language) solution*
- *Algorithmic method*

Simulation approach

- *The effect of distributions of the involved random variables on the distribution of the number of customers in the system*
- *The effect of distributions of the involved random variables on the mean and variance of the response/waiting time of a request*
- *The effect of distributions of the involved random variables on the mean and variance of the number of retrials*

Asymptotic method



Asymptotic of the first order

Let $Q(\infty)$ be the number of customers in the system in steady-state, then

$$\lim_{N\to\infty} E\exp\left\{iw\frac{Q(\infty)}{N}\right\} = \exp\left\{iw\kappa_1\right\} \ , \tag{1}$$

where the value of parameter κ_1 is the positive solution of the equation

$$\lambda\left(1-\kappa_1\right) - a[\kappa_1]\left[R_0[\kappa_1] - R_1[\kappa_1]\right] + \gamma_1 R_1[\kappa_1] = 0, \tag{2}$$

here $a\left[\kappa_1\right])$ is

$$a\left[\kappa_1\right] = \lambda\left(1-\kappa_1\right) + \sigma\kappa_1, \tag{3}$$

and the stationary distributions of probabilities $R_k[\kappa_1]$ of the service state k are defined as follows

$$R_0[\kappa_1] = \left\{\frac{\gamma_0+\gamma_2}{\gamma_2} + \frac{\gamma_1+\gamma_2}{\gamma_2}\cdot\frac{a\left[\kappa_1\right]}{a\left[\kappa_1\right]+\gamma_1}\left[1-B^*(a[\kappa_1]+\gamma_1)\right]\right\}^{-1},$$

$$R_1[\kappa_1] = R_0[\kappa_1]\frac{a\left[\kappa_1\right]}{a\left[\kappa_1\right]+\gamma_1}\cdot\left[1-B^*(a[\kappa_1]+\gamma_1)\right],$$

$$R_2[\kappa_1] = \frac{1}{\gamma_2}\left[\gamma_0 R_0[\kappa_1] + \gamma_1 R_1[\kappa_1]\right]. \tag{4}$$

Asymptotic of the second order

$$\lim_{N\to\infty} E\exp\left\{iw\frac{Q(\infty)-\kappa_1 N}{\sqrt{N}}\right\} = \exp\left\{\frac{(iw)^2}{2}\kappa_2\right\}, \qquad (5)$$

the value of parameter κ_2 is defined by expression

$$\kappa_2 = \frac{\lambda\,(1-\kappa_1)\left\{\, b_2\,[1-b_3] + \lambda\,(1-\kappa_1)\,(a+\gamma_1)R_2 + b_1\,[R_0-b_3]\right\}}{\lambda b_2 + (\sigma-\lambda)\left\{\,(b_1+b_2)\,[b_3 - R_1^*(a+\gamma_1)] - b_1\,[R_0-R_1]\right\}}, \qquad (6)$$

where

$$b_1 = \lambda\,(1-\kappa_1)\,(\gamma_1+\gamma_2), \quad b_2 = \gamma_2\,[a+\gamma_1]\,, \quad b_3 = R_0 B^*(a+\gamma_1).$$

From the proved theorem it follows that if $N \to \infty$ the limiting distributions for the centered and normalized number of customers in the system has a Gaussian distribution with variance κ_2, defined by the expression (6).

Corollary

As a consequence the distribution of the number of customers in the system is Gaussian with mean $N\kappa_1$ and variance $N\kappa_2$, respectively.

Generating function of the number of retrials

For the generating function of the number of transitions ν of the tagged customer into the orbit we have

$$\lim_{N\to\infty} \mathsf{E}\, z^{\nu} = \frac{1-q}{1-qz}, \qquad (7)$$

where the probability q can be obtained as

$$q = 1 - R_0 B^*(a+\gamma_1), \qquad (8)$$

Consequently, ν is geometric, namely

$$P\{\nu = n\} = (1-q)q^n, \quad n = \overline{0,\infty}, \tag{9}$$

and for the prelimit situation, that is when N is fixed we can and will use the following approximation $P\{\nu = n\} \approx (1-q)q^n$.

Characteristic function of the waiting time W of the tagged customer in the orbit

$$\mathsf{E}e^{iuW} \approx (1-q) + q\frac{\sigma(1-q)}{\sigma(1-q) - iuN}. \qquad (10)$$

The average sojourn time $\bar{T}_S$ of the customer under service

$$\bar{T}_S \approx \frac{1 - B^*(a+\gamma_1)}{(a+\gamma_1)B^*(a+\gamma_1)}. \tag{11}$$

Comparisons, examples

For the considered retrial queuing system we choose gamma distributed service time S with a shape parameter α and scale parameter β, with Laplace-Stieltjes transform $B^*(\delta)$ of the form

$$B^*(\delta) = \left(1 + \frac{\delta}{\beta}\right)^{-\alpha} ,$$

in the case when $\alpha = \beta$, that is when the average service time is equal to unit.

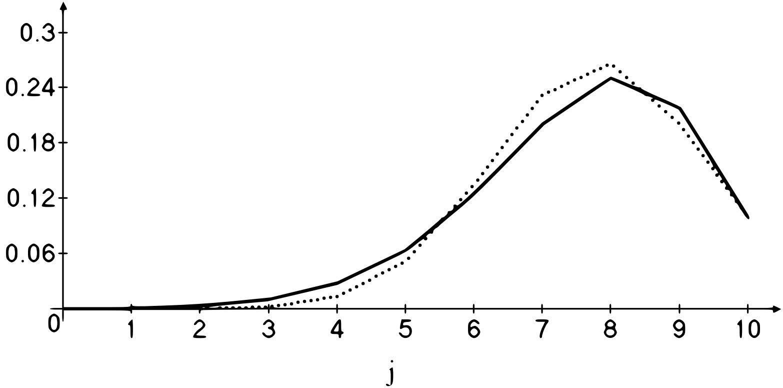
It can be shown that

$$\mathsf{E}(S) = \frac{\alpha}{\beta}, \quad Var(S) = \frac{\alpha}{\beta^2}, \quad V_S^2 = \frac{1}{\alpha},$$

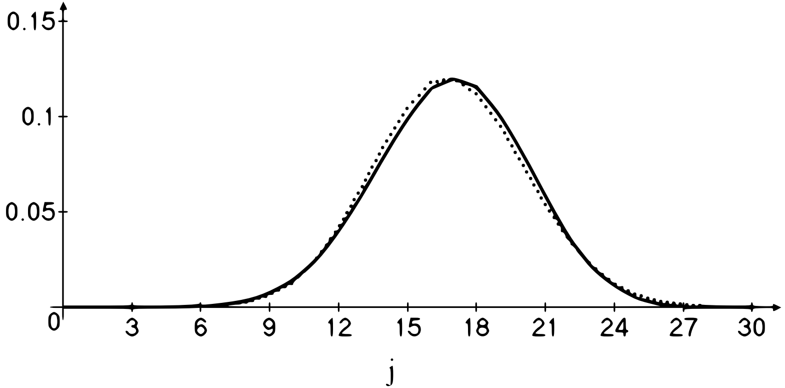
where V_S^2 denotes the squared coefficient of variation of S.

Input parameters:

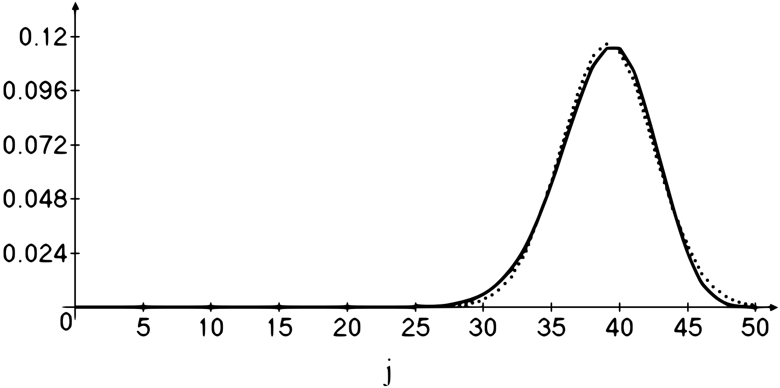
$$\lambda = 1 \ , \ \sigma = 5 \ , \ \gamma_0 = 0.1 \ , \ \gamma_1 = 0.2 \ , \ \gamma_2 = 1 \ , \ \alpha = \beta = 1.778$$



Comparison of the Gaussian approximation and numerical results in the case $N = 10$



Comparison of the Gaussian approximation and numerical results in the case $N = 30$



Comparison of the Gaussian approximation and numerical results in the case $N = 50$

To determine the accuracy and area of applicability of Gaussian approximation we will use the Kolmogorov-distance which can be defined as follows

$$\Delta = \max_{0 \leq j \leq N} \left| \sum_{n=0}^{j} \left(\Pi(n) - P_{as}(n) \right) \right| .$$

Table 1: Kolmogorov-distance between prelimit distribution $\Pi(j)$ and its normal approximation $P_{as}(j)$ for various values of N and $\alpha = \beta$

V		**5**	**10**	**20**	**30**	**50**	**100**
0,5	$\boldsymbol{\alpha = 4}$	0.090	0.068	0.039	0.030	0.024	0.018
0,75	$\boldsymbol{\alpha = 1,778}$	0.065	0.042	0.029	0.023	0.018	0.013
1	$\boldsymbol{\alpha = 1}$	0.037	0.023	0.017	0.014	0.011	0.007
1,5	$\boldsymbol{\alpha = 0,444}$	0.029	0.006	0.004	0.004	0.003	0.002
3	$\boldsymbol{\alpha = 0,111}$	0.010	0.068	0.030	0.019	0.016	0.014

Running the simulation program with inputs

$$\lambda = 1, \quad \sigma = 1, \quad \gamma_0 = 0.1, \quad \gamma_1 = 0.1, \quad \gamma_2 = 1,$$

and applying the proposed approximation (9) we calculate the Kolmogorov distance Δ for various values of N and $\alpha = \beta$ in Table 2.

Table 2: Kolmogorov distance between distribution of retrials $P_s(n)$ and $P_{as}(n)$ for various values of parameters N and $\alpha = \beta$

	$N = 10$	$N = 30$	$N = 50$	$N = 70$	$N = 100$
$\alpha = 0.5$	0.0218	0.0067	0.0038	0.0029	0.0021
$\alpha = 1$	0.0292	0.0099	0.0064	0.0048	0.0035
$\alpha = 2$	0.0360	0.0119	0.0075	0.0056	0.0040

Conclusions

1. Finite source retrial queueing system with collisions
2. Different solution approaches
3. Recent results on systems with an unreliable server
4. Graphical illustrations, comparisons, examples

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Thank You for Your Attention