Asymptotic analysis of the heterogeneous machine interference problem with random environments

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This paper presents a queueing model that can be used to analyze the asymptotic behavior of the machine interference problem composed of N *heterogeneous machines and* n *operatives. Each machine and the repair facility are assumed to operate in independent random environments governed by ergodic Markov chains. The running and repair times of a machine are assumed to be exponentially distributed random variables with parameters depending on the index of the machine and the state of the corresponding random environment. Assuming that the repair rates are much larger than the corresponding failure rates (i.e., "fast" service), it is shown that the time until the number of stopped machines reaches a certain level converges weakly, under appropriate norming, to an exponentially distributed random variable. Furthermore, some numerical examples that compare the approximate and exact characteristics are presented.*

Keywords: heterogeneous machine interference problem, random environment, failure-free operation time, weak convergence

Introduction

The machine interference problem has been considered by many authors over the last few decades by using a variety of approaches and making different assumptions about the statistical distributions of operating time between breakdowns and repair time. (Carmichael[1] and Stecke and Aronson[2] provide extensive bibliographies.) In recent years the machine interference model has been used, for example, for the mathematical description of computer terminal systems, cf., Takagi,[3] or for modelling production systems in textile winding, cf., Bunday.[4] More recently several authors have tackled the problem for nonidentical sets of machines. The major problem when considering different types of machines is that it is necessary to keep track of where each individual machine is in the system. Recent bibliographies on the heterogeneous machine problem include Bunday and Khorram,[5] Sztrik,[6,7] and Tosirisuk and Chandra.[8] In these papers, the main objective has been to predict steady-state operational measures, such as machine availability, operative utilization, mean waiting time, and average queue length. The diffusion approximation, cf., Sivazlian and Wang,[9] is based on the assumption that the queue of failed machines is almost always nonempty, i.e., we have a heavy traffic situation. In this study, another asymptotic approach is presented to analyze the distribution of the time until the number of stopped machines reaches a certain level. This method is quite common in reliability theory, e.g., Anisimov and Sztrik,[10] Gertsbakh,[11,12] and Keilson.[13]

Realistic consideration of certain stochastic systems, however, often requires the introduction of a random environment in which system parameters are subject to randomly occurring fluctuations. This situation may be attributed to certain changes in the physical environment or sudden personnel changes and workload alterations. Computational problems of birth-and-death models in random environments, sometimes called Markov-modulated processes, have been the subject of several works (c.f., Gaver et al.,[14] Neuts,[15,16] Purdue,[17] Sengupta,[18] and Stern and Elwalid[19]). Necessary and sufficient conditions for the stability of a single-server exponential queue with random fluctuations in the intensity of the arrival processes have also been derived (c.f., Baccelli and Makowski,[20] Gelenbe and Rosenberg,[21] and Rosenberg et al.[22]).

This paper presents a queueing model to analyze the asymptotic behavior of the machine interference problem with N heterogeneous machines and n operators. Each machine and the repair facility are assumed to operate in independent random environments governed

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by ergodic Markov chains. The running and repair times of a machine are supposed to be exponentially distributed random variables with parameters depending on the index of the machine and the state of the corresponding random environment. Assuming that the repair rates are much larger than the corresponding failure rates (i.e., "fast" service), it is shown that the time until the number of stopped machines reaches a certain level converges weakly, under appropriate norming, to an exponentially distributed random variable. Furthermore, some numerical examples are provided to illustrate various cases.

Preliminary results

This section presents a brief survey of results (c.f., Anisimov et al.[23]) to be applied in the next section.

Let $(X_\epsilon(k), k \geq 0)$ be a Markov chain depending on a small parameter $\epsilon > 0$ and let its state space be

$$\bigcup_{q=0}^{m+1} X_q, \text{ where } X_i \cap X_j = \emptyset, i \neq j, \quad \text{for } i,j = 0, 1, \ldots, m+1 \quad (1)$$

It is assumed that the transition matrix $(p_\epsilon(i^{(q)}, j^{(z)}))$, $i^{(q)} \in X_q$, $j^{(z)} \in X_z$, $q, z = 0, 1, \ldots, m+1$ satisfies the following conditions:

1. $p_\epsilon(i^{(0)}, j^{(0)}) \to p_0(i^{(0)}, j^{(0)})$, as $\epsilon \to 0$, for $i^{(0)}, j^{(0)} \in X_0$, and matrix $P_0 = (p_0(i^{(0)}, j^{(0)}))$ is irreducible;
2. $p_\epsilon(i^{(q)}, j^{(q+1)}) = \epsilon\alpha^{(q)}(i^{(q)}, j^{(q+1)}) + o(\epsilon)$, for $i^{(q)} \in X_q$, $j^{(q+1)} \in X_{q+1}$, where $\alpha^{(q)}(i^{(q)}, j^{(q+1)})$ is an appropriate transition matrix;
3. $p_\epsilon(i^{(q)}, f^{(q)}) \to 0$, as $\epsilon \to 0$, for $i^{(q)}, f^{(q)} \in X_q$, $q \geq 1$;
4. $p_\epsilon(i^{(q)}, f^{(z)}) \equiv 0$, for $i^{(q)} \in X_q$, $f^{(z)} \in X_z$, $z - q \geq 2$.

In the sequel the set of states X_q is called the qth level of the chain, $q = 0, \ldots, m+1$. Let us single out the subset of states

$$\langle \alpha_m \rangle = \bigcup_{q=0}^{m} X_q, \text{ where } X_i \cap X_j = \phi, i \neq j, \quad \text{for } i, j = 0, 1, \ldots, m \quad (2)$$

Denote by $\{\pi_\epsilon(i^{(q)}), i^{(q)} \in X_q\}$, $q = 1, \ldots, m$ the stationary distribution of a chain with transition matrix

$$\left(\frac{p_\epsilon(i^{(q)}, j^{(z)})}{1 - \sum_{k^{(m+1)} \in X_{m+1}} p_\epsilon(i^{(q)}, k^{(m+1)})} \right), \quad i^{(q)} \in X_q, j^{(z)} \in X_z, q, z \leq m \quad (3)$$

Furthermore denote by $g_\epsilon(\langle \alpha_m \rangle)$ the steady-state probability of exit from $\langle \alpha_m \rangle$; that is

$$g_\epsilon(\langle \alpha_m \rangle) = \sum_{i^{(m)} \in X_m} \pi_\epsilon(i^{(m)}) \sum_{j^{(m+1)} \in X_{m+1}} p_\epsilon(i^{(m)}, j^{(m+1)}) \quad (4)$$

Denote by $\{\pi_o(i^{(0)}), i^{(0)} \in X_0\}$ the stationary distribution corresponding to P_o and let

$$\bar{\pi}_o = \{\pi_o(i^{(0)}), i^{(0)} \in X_0\}, \quad \bar{\pi}_\epsilon^{(q)} = \{\pi_\epsilon(i^{(q)}), i^{(q)} \in X_q\} \quad (5)$$

be row vectors. Finally, let the matrix

$$A^{(q)} = (\alpha^{(q)}(i^{(q)}, j^{(q+1)})), \quad i^{(q)} \in X_q, j^{(q+1)} \in X_{q+1}, \quad q = 0, \ldots, m \quad (6)$$

defined by condition 2.

Conditions 1–4 enable us to compute the main terms of the asymptotic expression for $\bar{\pi}_\epsilon^{(q)}$ and $g_\epsilon(\langle \alpha_m \rangle)$. Namely, we obtain

$$\bar{\pi}_\epsilon^{(q)} = \epsilon^q \bar{\pi}_o A^{(0)} A^{(1)} \ldots A^{(q-1)} + o(\epsilon^q), \quad q = 1, \ldots, m, \quad (7)$$
$$g_\epsilon(\langle \alpha_m \rangle) = \epsilon^{m+1} \bar{\pi}_o A^{(0)} A^{(1)} \ldots A^{(m)} \underline{1} + o(\epsilon^{m+1})$$

where $\underline{1} = (1, \ldots, 1)^*$ is a column vector (c.f., Anisimov et al.,[23] pp. 141–153). Let $(\eta_\epsilon(t), t \geq 0)$ be a semi-Markov process (SMP) given by the embedded Markov chain $(X_\epsilon(k), k \geq 0)$ satisfying conditions 1–4. Let the times $\tau_\epsilon(j^{(s)}, k^{(z)})$ be transition times from state $j^{(s)}$ to state $k^{(z)}$ that fulfil the condition

$$E \exp\{i\theta\beta_\epsilon\tau_\epsilon(j^{(s)}, k^{(z)})\} = 1 + a_{j^{(s)}k^{(z)}}(\theta)\epsilon^{m+1} + o(\epsilon^{m+1}), \quad (i^2 = -1) \quad (8)$$

where β_ϵ is some normalizing factor. Denote by $\Omega_\epsilon(m)$ the instant at which the SMP reaches the $(m+1)$th level for the first time, exit time from $\langle \alpha_m \rangle$, provided $\eta_\epsilon(0) \in \langle \alpha_m \rangle$. Then we have

Theorem 1. (c.f., Anisimov et al.,[23] p. 153) If conditions 1–4 are satisfied then

$$\lim_{\epsilon \to} E \exp\{i\theta\beta_\epsilon\Omega_\epsilon(m)\} = (1 - A(\theta))^{-1} \quad (9)$$

where

$$A(\theta) = \frac{\sum_{j^{(0)}, k^{(0)} \in X_o} \pi_o(j^{(0)}) p_o(j^{(0)}, k^{(0)}) a_{j^{(0)}k^{(0)}}(\theta)}{\bar{\pi}_o A^{(0)} A^{(1)} \ldots A^{(m)} \underline{1}} \quad (10)$$

Corollary 1. In particular, if $a_{j^{(s)}k^{(z)}}(\theta) = i\theta m_{j^{(s)}k^{(z)}}$, then the limit is an exponentially distributed random variable with mean

$$\frac{\sum_{j^{(0)}, k^{(0)} \in X_o} \pi_o(j^{(0)}) p_o(j^{(0)}, k^{(0)}) m_{j^{(0)}k^{(0)}}}{\bar{\pi}_o A^{(0)} A^{(1)} \ldots A^{(m)} \underline{1}} \quad (11)$$

The queueing model

Let us consider the machine interference problem with N heterogeneous machines supervised by n operatives of the same kind. Machine p is assumed to operate in a random environment governed by an ergodic Markov chain $(\xi_p(t), t \geq 0)$ with state space $(1, \ldots, r_p)$ and with transition density matrix $(a_{i_pj_p}^{(p)}, i_p, j_p = 1, \ldots, r_p, a_{i_pi_p}^{(p)} = \sum_{k \neq i_p} a_{i_pk}^{(p)})$. Whenever the environmental process $\xi_p(t)$ is in state i_p, the probability that machine p breaks down in the time interval $(t, t+h)$ is $\lambda_p(i_p)h + o(h)$, $p = 1, \ldots, N$. Each machine is immediately repaired if there is an idle operative; otherwise a queueing line is formed. The service discipline is first come-first served (FCFS). The repair facility is also supposed to operate in a random environment governed by an ergodic

Markov chain $(\xi_{N+1}(t), t \geq 0)$ with state space $(1, \ldots, r_{N+1})$ and with transition density matrix $(a^{(N+1)}_{i_{N+1}j_{N+1}}, i_{N+1}, j_{N+1} = 1, \ldots, r_{N+1}, a^{(N+1)}_{i_{N+1}i_{N+1}} = \Sigma_{k \neq i_{N+1}} a^{(N+1)}_{i_{N+1}k})$. Whenever the environmental process $\xi_{N+1}(t)$ is in state i_{N+1} and there are s machines stopped, $s = 1, \ldots, N$, the probability that the repair of machine p is completed in time interval $(t, t+h)$ is $\mu_p(i_{N+1}, s; \epsilon)h + o(h)$. After being repaired each machine immediately starts operating. All random variables involved here and the random environments are supposed to be independent of each other.

Let us consider the system under the assumption of "fast" repair, i.e., $\mu_p(i_{N+1}, s; \epsilon) \to \infty$ as $\epsilon \to 0$. For simplicity let $\mu_p(i_{N+1}, s; \epsilon) = \mu_p(i_{N+1}, s)/\epsilon$.

Denote by $Y_\epsilon(t)$ the number of stopped machines at time t, and let

$$\Omega_\epsilon(m) = \inf\{t: t>0, Y_\epsilon(t) = m + 1 / Y_\epsilon(0) \leq m\} \quad (12)$$

that is, the instant at which the number of stopped machines reaches the $(m+1)$th level for the first time, provided that at the beginning their number is not greater than m; $m = 1, \ldots, N-1$.

Denote by $(\pi^{(p)}_{i_p}, i_p = 1, \ldots, r_p)$ the steady-state distribution of the governing Markov chains $(\xi_p(t), t \geq 0)$, $p = 1, \ldots, N+1$, respectively, and let V^s_N be the set of all variations of order s of integers $1, \ldots, N$. Then we have:

Theorem 2. For the system under the above assumptions, independently of the initial state, the distribution of the normalized random variable $\epsilon^m \Omega_\epsilon(m)$ converges weakly to an exponentially distributed random variable with parameter

$$\Lambda = \sum_{i_1=1}^{r_1} \cdots \sum_{i_{N+1}=1}^{r_{N+1}} \sum_{(k_1,\ldots,k_{m+1}) \in V^{m+1}_N} \left(\prod_{p=1}^{N+1} \pi^{(p)}_{i_p}\right) \frac{\prod_{s=0}^{m} \lambda_{k_{s+1}}(i_{k_{s+1}})}{\prod_{s=1}^{m} \sum_{q=1}^{\min(s,n)} \mu_{k_q}(i_{N+1}, s)} \quad (13)$$

Proof. It can be seen that the process

$$Z_\epsilon(t) = (\xi_1(t), \ldots, \xi_{N+1}(t): Y_\epsilon(t); \gamma_1(t), \ldots, \gamma_{Y_\epsilon(t)}(t)) \quad (14)$$

is a multidimensional Markov chain with state space

$$((i_1, \ldots, i_{N+1}: s; k_1, \ldots k_s), i_p = 1, \ldots, r_p, \; p = 1, \ldots, N+1, (k_1, \ldots k_s) \in V^s_N, s = 0, \ldots, N) \quad (15)$$

where $\gamma_1(t), \ldots, \gamma_{Y_\epsilon(t)}(t)$ denote the indices of failed machines at time t in the order of their breakdowns and by definition $k_o = \{0\}$. Furthermore, let

$$\langle \alpha_m \rangle = ((i_1, \ldots, i_{N+1}: s; k_1, \ldots k_s), i_p = 1, \ldots, r_p, \; p = 1, \ldots, N+1, (k_1, \ldots k_s) \in V^s_N, s = 0, \ldots, m) \quad (16)$$

Hence our aim is to determine the distribution of the first exit time of $Z_\epsilon(t)$ from $\langle \alpha_m \rangle$, provided that $Z_\epsilon(o) \in \langle \alpha_m \rangle$.

It can readily be verified that the transition probabilities in any time interval $(t, t+h)$ are the following:

$$(i_1, \ldots, i_{N+1}: s; k_1, \ldots, k_s) \to h$$

$$\begin{cases} (i_1, \ldots, j_p, \ldots, i_{N+1}: s; k_1, \ldots, k_s) & a^{(p)}_{i_p j_p} h + o(h), s = 0, \ldots, N, j_p \neq i_p, p = 1, \ldots, N+1 \\ (i_1, \ldots, i_{N+1}: s+1; k_1, \ldots, k_{s+1}) & \lambda_{k_{s+1}}(i_{k_{s+1}})h + o(h), s = 0, \ldots, N-1 \\ (i_1, \ldots, i_{N+1}: s-1; k_1, \ldots, k_{q-1}, k_{q+1}, \ldots, k_s) & \mu_{k_q}(i_{N+1}, s)h/\epsilon + o(h), s = 1, \ldots, N, q = 1, \ldots, \min(s,n) \end{cases} \quad (17)$$

In addition, the sojourn time $\tau_\epsilon(i_1, \ldots, i_{N+1}: s; k_1, \ldots, k_s)$ of $Z_\epsilon(t)$ in state $(i_1, \ldots, i_{N+1}: s; k_1, \ldots, k_s)$ is exponentially distributed with parameter

$$a^{(1)}_{i_1 i_1} + \cdots + a^{(N+1)}_{i_{N+1} i_{N+1}} + \sum_{j \neq k_1, \ldots, k_s}^{N} \lambda_j(i_j) + \sum_{q=1}^{\min(s,n)} \mu_{k_q}(i_{N+1}, s)/\epsilon \quad (18)$$

where by definition $\mu_{k_q}(i_{N+1}, 0) = 0$. Thus, the transition probabilities for the embedded Markov chain are

$$p_\epsilon[(i_1, \ldots, i_{N+1}: s; k_1, \ldots, k_s), (i_1, \ldots, j_p, \ldots, i_{N+1}: s; k_1, \ldots, k_s)] = \frac{a^{(p)}_{i_p j_p}}{a^{(1)}_{i_1 i_1} + \cdots + a^{(N+1)}_{i_{N+1} i_{N+1}} + \sum_{j \neq k_1, \ldots, k_s} \lambda_j(i_j) + \sum_{q=1}^{\min(s,n)} \mu_{k_q}(i_{N+1}, s)/\epsilon}$$

$$\text{for} \quad s = 0, \ldots, N, p = 1, \ldots, N+1 \quad (19)$$

$$p_\epsilon[(i_1, \ldots, i_{N+1}: s; k_1, \ldots, k_s), (i_1, \ldots, i_{N+1}: s+1; k_1, \ldots, k_{s+1})] = \frac{\lambda_{k_{s+1}}(i_{k_{s+1}})}{a^{(1)}_{i_1 i_1} + \cdots + a^{(N+1)}_{i_{N+1} i_{N+1}} + \sum_{j \neq k_1, \ldots, k_s} \lambda_j(i_j) + \sum_{q=1}^{\min(s,n)} \mu_{k_q}(i_{N+1}, s)/\epsilon}$$

$$\text{for} \quad s = 0, \ldots, N-1 \quad (20)$$

$$p_\epsilon[(i_1,\ldots,i_{N+1}:s;k_1,\ldots,k_s),(i_1,\ldots,i_{N+1}:s-1;k_1,\ldots,k_{q-1},k_{q+1},\ldots,k_s)] = \frac{\mu_{k_q}(i_{N+1},s)/\epsilon}{a^{(1)}_{i_1i_1}+\cdots+a^{(N+1)}_{i_{N+1}i_{N+1}}+\sum_{j\neq k_1,\ldots,k_s}\lambda_j(i_j)+\sum_{q=1}^{\min(s,n)}\mu_{k_q}(i_{N+1},s)/\epsilon}$$

$$\text{for}\quad q=1,\ldots,\min(s,n), s=1,\ldots,N \quad (21)$$

Notice that equations (19), (20), and (21) mean that one of the random environments changes its state, machine $i_{k_{s+1}}$ breaks down, and the repair of machine k_q is completed, respectively. That is why the number of stopped machines is s, $s+1$, and $s-1$, respectively.

As $\epsilon\to 0$ equations (19), (20), and (21) imply, respectively, the following

$$p_\epsilon[(i_1,\ldots,i_{N+1}:0;0),(i_1,\ldots,j_p,\ldots,i_{N+1}:0;0)] = \frac{a^{(p)}_{i_pj_p}}{a^{(1)}_{i_1i_1}+\cdots+a^{(N+1)}_{i_{N+1}i_{N+1}}+\sum_{j=1}^{N}\lambda_j(i_j)},$$

$$p=1,\ldots,N+1, s=0 \quad (22)$$

$$p_\epsilon[(i_1,\ldots,i_{N+1}:s;k_1,\ldots,k_s),(i_1,\ldots,j_p,\ldots,i_{N+1}:s;k_1,\ldots,k_s)] = o(1) \quad \text{for } s=1,\ldots,N \quad (23)$$

$$p_\epsilon[(i_1,\ldots,i_{N+1}:0;0),(i_1,\ldots,i_{N+1}:1;k)] = \frac{\lambda_k(i_k)}{a^{(1)}_{i_1i_1}+\cdots+a^{(N+1)}_{i_{N+1}i_{N+1}}+\sum_{j=1}^{N}\lambda_j(i_j)} \quad (24)$$

$$p_\epsilon[(i_1,\ldots,i_{N+1}:s;k_1,\ldots,k_s),(i_1,\ldots,i_{N+1}:s+1;k_1,\ldots,k_{s+1})] = \frac{\lambda_{k_{s+1}}(i_{k_{s+1}})\epsilon}{\sum_{q=1}^{\min(s,n)}\mu_{k_q}(i_{N+1},s)}(1+o(1)), s=1,\ldots,N-1 \quad (25)$$

This agrees with conditions 1–4, but here the zero level is the set $((i_1,\ldots,i_{N+1}:0;0),(i_1,\ldots,i_{N+1}:1;k), i_p=1,\ldots,r_p, p=1,\ldots,N+1, k=1,\ldots,N)$, while the qth level is the set $((i_1,\ldots,i_{N+1}:q+1;k_1,\ldots,k_{q+1}), i_p=1,\ldots,r_p, p=1,\ldots,N+1, (k_1,\ldots,k_{q+1})\in V_N^{q+1})$. Because the level 0 in the limit forms an essential class, the probabilities $\pi_0(i_1,\ldots,i_{N+1}:0;0)$, $\pi_0(i_1,\ldots,i_{N+1}:1;k), i_p=1,\ldots,r_p, p=1,\ldots,N+1, k=1,\ldots,N$ satisfy the following system of equations:

$$\pi_o(j_1,\ldots,j_{N+1}:0;0) = \sum_{i_1\neq j_1}\pi_o(i_1,j_2,\ldots,j_{N+1}:0;0)\,x a^{(1)}_{i_1j_1}(a^{(1)}_{i_1i_1}+a^{(2)}_{j_2j_2}+\cdots+a^{(N+1)}_{j_{N+1}j_{N+1}}+\lambda_1(i_1)+\lambda_2(j_2)+\cdots+\lambda_N(j_N))^{-1} + \sum_{i_2\neq j_2}\pi_o(j_1,i_2,\ldots,j_{N+1}:0;0)\,x a^{(2)}_{i_2j_2}(a^{(1)}_{j_1j_1}+a^{(2)}_{i_2j_2}+\cdots+a^{(N+1)}_{j_{N+1}j_{N+1}}+\lambda_1(j_1)+\lambda_2(i_2)+\cdots+\lambda_N(j_N))^{-1}+\cdots+\sum_{i_{N+1}\neq j_{N+1}}\pi_o(j_1,j_2,\ldots,i_{N+1}:0;0)\,x a^{(N+1)}_{i_{N+1}j_{N+1}}(a^{(1)}_{j_1j_1}+a_{j_2j_2}+\cdots+a^{(N+1)}_{i_{N+1}j_{N+1}}+\lambda_1(j_1)+\lambda_2(j_2)+\cdots+\lambda_N(i_N))^{-1}+\pi_o(j_1,\ldots,j_{N+1}:1;k) \quad (26)$$

$$\pi_o(j_1,\ldots,j_{N+1}:1;k) = \pi_o(j_1,\ldots,j_{N+1}:0;0)\lambda_k(j_k)\left(a^{(1)}_{j_1j_1}+\cdots+a^{(N+1)}_{j_{N+1}j_{N+1}}+\sum_{k=1}^{N}\lambda_k(j_k)\right)^{-1} \quad (27)$$

It follows that

$$\pi^{(p)}_{j_p}a^{(p)}_{j_pj_p} = \sum_{i_p\neq j_p}\pi^{(p)}_{i_p}a^{(p)}_{i_pj_p}, p=1,\ldots,N+1 \quad (28)$$

It can be verified that the solution of (26), (27), with (28) is

$$\pi_o(i_1,\ldots,i_{N+1}:0;0) = B(\pi^{(1)}_{i_1}\cdots\pi^{(N+1)}_{i_{N+1}})\left(a^{(1)}_{i_1i_1}+\cdots+a^{(N+1)}_{i_{N+1}i_{N+1}}+\sum_{k=1}^{N}\lambda_k(i_k)\right) \quad (29)$$

$$\pi_o(i_1,\ldots,i_{N+1}:1;k) = B(\pi^{(1)}_{i_1}\cdots\pi^{(N+1)}_{i_{N+1}})\lambda_k(i_k) \quad (30)$$

where B is the normalizing constant, i.e.,

$$1/B = \sum_{i_1=1}^{r_1}\cdots\sum_{i_{N+1}=1}^{r_{N+1}}(\pi^{(1)}_{i_1}\cdots\pi^{(N+1)}_{i_{N+1}})\left(a^{(1)}_{i_1i_1}+\cdots+a^{(N+1)}_{i_{N+1}i_{N+1}}+2\sum_{k=1}^{N}\lambda_k(i_k)\right) \quad (31)$$

By using equation (7) it can be shown that the probability of exit from $\langle \alpha_m \rangle$ is

$$B\epsilon^m \sum_{i_1=1}^{r_1} \cdots \sum_{i_{N+1}=1}^{r_{N+1}} \sum_{(k_1,\ldots,k_{m+1})\in V_N^{m+1}} \left(\prod_{p=1}^{N+1} \pi_{i_p}^{(p)} \right) \frac{\prod_{s=0}^{m} \lambda_{k_{s+1}}(i_{k_{s+1}})}{\prod_{s=1}^{m} \sum_{q=1}^{\min(s,n)} \mu_{k_q}(i_{N+1}, s)} (1 + o(1)) \tag{32}$$

Taking into account the exponentiality of $\tau_\epsilon(i_1, \ldots, i_{N+1}:s; k_1, \ldots, k_s)$ for fixed θ we have

$$E\exp\{i\epsilon^m\theta\tau_\epsilon(j_1, \ldots, j_{N+1}:0;0)\} = 1 + \epsilon^m \frac{i\theta}{a_{j_1 j_1}^{(1)} + \cdots + a_{j_{N+1} j_{N+1}}^{(N+1)} + \sum_{k=1}^{N} \lambda_k(j_k)} (1 + o(1)) \tag{33}$$

$$E\exp\{i\epsilon^m\theta\tau_\epsilon(j_1, \ldots, j_{N+1}:s;k_1, \ldots, k_s)\} = 1 + o(\epsilon^m), s > 0 \tag{34}$$

Notice that $\beta_\epsilon = \epsilon^m$ and therefore from Corollary 1 we immediately get the statement that $\epsilon^m\Omega_\epsilon(m)$ converges weakly to an exponentially distributed random variable with parameter given by

$$\Lambda = \sum_{i_1=1}^{r_1} \cdots \sum_{i_{N+1}=1}^{r_{N+1}} \sum_{(k_1,\ldots,k_{m+1})\in V_N^{m+1}} \left(\prod_{p=1}^{N+1} \pi_{i_p}^{(p)} \right) \frac{\prod_{s=0}^{m} \lambda_{k_{s+1}}(i_{k_{s+1}})}{\prod_{s=1}^{m} \sum_{q=1}^{\min(s,n)} \mu_{k_q}(i_{N+1}, s)} \tag{35}$$

which completes the proof.

Consequently, the distribution of the time until the number of stopped machines reaches the $(m + 1)$th level for the first time can be approximated by

$$P(\Omega_\epsilon(m) > t) = P(\epsilon^m\Omega_\epsilon(m) > \epsilon^m t) \approx \exp(-\epsilon^m \Lambda t) \tag{36}$$

i.e., $\Omega_\epsilon(m)$ is asymptotically an exponentially distributed random variable with parameter $\epsilon^m\Lambda$.

In particular, for $m = N - 1$, which means that there is no operating machine, we have

$$\epsilon^{N-1}\Lambda = \epsilon^{N-1} \sum_{i_1=1}^{r_1} \cdots \sum_{i_{N+1}=1}^{r_{N+1}} \sum_{(k_1,\ldots,k_N)\in V_N^N} \left(\prod_{p=1}^{N+1} \pi_{i_p}^{(p)} \right) \frac{\prod_{s=0}^{N-1} \lambda_{k_{s+1}}(i_{k_{s+1}})}{\prod_{s=1}^{N-1} \sum_{q=1}^{\min(s,n)} \mu_{k_q}(i_{N+1}, s)} \tag{37}$$

In the case, where there are no random environments from (37) we get

$$\Lambda^* = \epsilon^{N-1}\Lambda = \epsilon^{N-1} \sum_{(k_1,\ldots,k_N)\in V_N^N} \frac{\prod_{s=0}^{N-1} \lambda_{k_{s+1}}}{\prod_{s=1}^{N-1} \sum_{q=1}^{\min(s,n)} \mu_{k_q}(s)} \tag{38}$$

where $\lambda_p(i_p) = \lambda_p$, $i_p = 1, \ldots, r_p$, $\mu_p(i_{N+1}, s) = \mu_p(s)$, $i_{N+1} = 1, \ldots, r_{N+1}$, $p = 1, \ldots, N$. Furthermore, if each machine has the same repair rate μ, then (38) yields

$$\Lambda^* = \epsilon^{N-1}\Lambda = \frac{N!}{n!n^{N-n-1}} \frac{\lambda_1 \cdots \lambda_N}{(\mu/\epsilon)^{N-1}} \tag{39}$$

Finally, for homogeneous failure rates from (39) we have

$$\Lambda^* = \epsilon^{N-1}\Lambda = \frac{N!}{n!n^{N-n-1}} \frac{\lambda^N}{(\mu/\epsilon)^{N-1}} \tag{40}$$

Hence, by using (37) the steady-state probability Q_W that at least one machine works is

$$Q_W = \frac{1}{\epsilon^{N-1}\Lambda} \left(\frac{1}{\epsilon^{N-1}\Lambda} + B_N \right)^{-1} \tag{41}$$

where B_N denotes the mean period of time during which all machines are stopped. And it can be shown from (39), that we can simplify (41) to obtain

$$Q_W = \left(1 + \frac{N!}{n!n^{N-n}} \frac{\lambda_1 \cdots \lambda_N}{(\mu/\epsilon)^N} \right)^{-1} \tag{42}$$

Finally, for the simplest homogeneous case we have

$$Q_W = 1/\left(1 + \frac{N!}{n!n^{N-n}} \left(\frac{\lambda}{\mu/\epsilon} \right)^N \right) \tag{43}$$

Some numerical results

In this section some numerical examples are given to illustrate the effectiveness of the method proposed by comparing the approximate results with the exact ones.

Case 1. Here $\rho = \lambda\epsilon/\mu$, and the exact steady-state probability P_W that at least one machine works is $P_W =$

$1 - (N!/n!n^{N-n})\,\rho^N P_o$ (via the well-known Palm formula, cf. Stecke and Aronson[2] or Bunday[4]). With $n = 3$ by using (40) and (43) we get the results shown in *Table 1*.

We can see how Q_W depends on N, ρ and how accurate it is. It should be noted that the greater the value of N the less the value of ρ for an acceptable approximation. Furthermore, we can observe the sharp increase of $1/\Lambda^*$, which is clearly the mean duration of the failure-free operation time of the system.

Case 2. In this section we deal with heterogeneous failure and homogeneous repair rates. By using (39) and (42) we get the asymptotic results against P_W derived by Sztrik.[6] *Table 2* shows the result using the following parameters: $N = 4$, $n = 2$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$, and $\lambda_4 = 4$. From the above examples we can conclude that the characteristics definitely depend on the repair rate μ/ϵ.

Conclusion

It should be noted that similar weak convergence arguments can be found in Keilson[13] for Markov chains that assume the steady-state distribution of the underlying process. This assumption can be difficult to obtain for the research presented in this paper because of the large number of states. The main objective of this study is an asymptotic approach providing a simple formula for the steady-state characteristics of the system.

Table 1. Numerical results relating to Case 1.

ρ	P_W	Q_W	$1/\Lambda^*$
		$N = 5$	
1	0.936305732	0.310344828	0.1
2^{-1}	0.991023339	0.935064935	2.4
2^{-2}	0.999290680	0.997834560	38.4
2^{-3}	0.999962376	0.999932188	614.4
2^{-4}	0.999998435	0.999997881	9830.4
2^{-5}	0.999999943	0.999999934	157286.4
2^{-6}	0.999999998	0.999999998	2516582.4
2^{-7}	1	1	40265318.4
		$N = 15$	
1	0.950212859	2.43840 E-6	8.1280 E-7
2^{-1}	0.997516417	7.39898 E-2	1.3316 E-2
2^{-2}	0.999991894	0.999618207	218.1
2^{-3}	0.999999998	0.999999988	3574746.5
2^{-4}	1	1	5.856864 E10
		$N = 25$	
1	0.950212932	1.2138 E-14	4.0462 E-15
2^{-1}	0.997521248	4.07307 E-7	6.78846 E-8
2^{-2}	0.999993850	0.93181969	1.1
2^{-3}	1	0.99999999	19107840.5
2^{-4}	1	1	3.20576 E20

Table 2. Numerical results relating to Case 2.

μ/ϵ	P_W	Q_W	$1/\Lambda^*$
1	0.626943005	1.36986 E-2	6.94444 E-3
5	0.977478886	0.896700144	0.8
10	0.997039717	0.992851469	6.9
20	0.999718279	0.999550202	55.5
30	0.999935358	0.999911119	187.5

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