Some Features of a Finite-Source M/GI/1 Retrial Queuing System with Collisions of Customers

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Abstract. In this paper a finite-source M/GI/1 retrial queuing system with collisions of customers is considered. The definition of throughput of the system as average number of customers, which are successfully served per unit time is introduced. It is shown that at some combinations of system parameter values and probability distribution of service time of customers the throughput can be arbitrarily small, and at another values of parameters throughput can be greater than the service intensity. Applying method of asymptotic analysis under the condition of unlimited growing number of sources it is proofed that limiting distribution of the number of retrials/transitions of the customer into the orbit is geometric and the sojourn/waiting time of the customer in the orbit follows a generalized exponential distribution. In addition, the mean sojourn time of the customer under service is obtained.

Keywords: Closed queuing system · Finite-source queuing system · Retrial queue · Collisions · Asymptotic analysis · Throughput · Sojourn time · Limiting distribution · Number of transitions into an orbit

1 Introduction

Retrial queues have been widely used to model many problems arising in telephone switching systems, telecommunication networks, computer networks and systems, call centers, etc. In many practical situations it is important to take into account the fact that the rate of generation of new primary calls decreases as the number of customers in the system increases. This can be done with the help of finite-source, or quasi-random input models, see, for example [1,2,7].

Another very important component of queuing models is the collisions of the customers. In the main model it is assumed that if an arriving customer finds

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the server busy, it involves into collision with customer under service and they both moves into the orbit. See, for example [3,4,6].

The aim of the present paper is to investigate such systems which has the above mention properties, that is finite-source, retrial and collisions of customer in the case of non-Markov service. This article paper is a continuation of the paper [5] in which the asymptotic distribution of the number of customers in the system was investigated.

2 Model Description and Notations

Let us consider a closed retrial queuing system of type M/GI/1//N with collision of the customers. The number of sources is N and each of them can generate a primary request during an exponentially distributed time with rate λ/N. A source cannot generate a new call until end of the successful service of this customer. If a primary customer finds the server idle, he enters into service immediately, in which the required service time has a probability distribution function $B(x)$. Let us denote its hazard rate function by $\mu(y) = B^{'}(y)(1-B(y))^{-1}$ and Laplace -Stieltjes transform by $B^{*}(y)$, respectively. If the server is busy, an arriving (primary or repeated) customer involves into collision with customer under service and they both moves into the orbit. The retrial time of requests are exponentially distributed with rate σ/N. All random variables involved in the model construction are assumed to be independent of each other.

The system state at time t is denoted by $\{l(t), i(t), y(t)\}$, where $i(t)$ is the number of customers in the system at time t, that is, the total number of customers in orbit and in service, $l(t)$ describes the server state as follows

$$l(t) = \begin{cases} 0, & \text{if the server is free,} \\ 1, & \text{if the server is busy,} \end{cases}$$

$y(t)$ is the supplementary random variable, equal to the elapsed service time of the customer till the moment t.

Thus, we will investigate the Markov process $\{l(t), i(t), y(t)\}$, which has a variable number of components, depending on the server state, since the component $y(t)$ is determined only in those moments when $l(t) = 1$.

Let us define the probabilities as follows:

$$p_0(i,t) = P\{l(t) = 0, i(t) = i\},$$
$$p_1(i,y,t) = \frac{\partial P\{l(t) = 1, i(t) = i, y(t) < y\}}{\partial y}.$$

Assuming that system is operating in steady state, for the stationary probability distribution $p_0(i)$, $p_1(i,y)$ by using standard methods the following system of Kolmogorov equations can be derived

$$
\begin{aligned}
&-\left[\frac{N-i}{N}\lambda+\frac{i}{N}\sigma\right]p_0(i)+\int\limits_0^\infty p_1(i+1,y)\mu(y)dy\\
&\qquad\qquad+\left(1-\frac{i-1}{N}\right)\lambda p_1(i-1)+\frac{i-1}{N}\sigma p_1(i)=0,\\
&\frac{\partial p_1(i,y)}{\partial y}=-\left[\frac{N-i}{N}\lambda+\frac{i-1}{N}\sigma+\mu(y)\right]p_1(i,y),
\end{aligned}
\tag{1}
$$

with boundary conditions

$$
p_1(i,0)=\left(1-\frac{i-1}{N}\right)\lambda p_0(i-1)+\frac{i}{N}\sigma p_0(i).
\tag{2}
$$

Denote the partial characteristic functions

$$
H_0(u)=\sum_{i=0}^{N}e^{jui}p_0(i)\;,\qquad H_1(u,y)=\sum_{i=1}^{N}e^{jui}p_1(i,y)\;,
\tag{3}
$$

where $j=\sqrt{-1}$ is imaginary unit, then system (1) and Eq. (2) can be rewritten as

$$
\begin{aligned}
&-\lambda H_0(u)+\left[\lambda e^{ju}-\frac{\sigma}{N}\right]H_1(u)+e^{-ju}\int\limits_0^\infty H_1(u,y)\mu(y)dy\\
&\qquad\qquad+j\frac{(\sigma-\lambda)}{N}\frac{dH_0(u)}{du}+j\frac{(\lambda e^{ju}-\sigma)}{N}\frac{dH_1(u)}{du}=0\;,\\
&\frac{\partial H_1(u,y)}{\partial y}=\left[\frac{\sigma}{N}-\lambda-\mu(y)\right]H_1(u,y)-j\frac{(\lambda-\sigma)}{N}\frac{\partial H_1(u,y)}{\partial u},\\
&H_1(u,0)=\lambda e^{ju}H_0(u)+j\frac{(\lambda e^{ju}-\sigma)}{N}\frac{dH_0(u)}{du}.
\end{aligned}
\tag{4}
$$

3 Asymptotic Analysis

By using asymptotic methods [8] as a consequence of the first order solution to (4) can be obtained as follows

Theorem 1. *Let $i(t)$ be number of customers in a closed retrial queuing system M/GI/1//N with the collisions of customers, then*

$$
\lim_{N\to\infty}\mathsf{E}\exp\left\{jw\frac{i(t)}{N}\right\}=\exp\left\{jw\kappa\right\},
\tag{5}
$$

where value of parameter κ is the positive solution of the equation

$$
(1-\kappa)\,\lambda-\delta(\kappa)\frac{B^*(\delta(\kappa))}{2-B^*(\delta(\kappa))}=0,
\tag{6}
$$

here $\delta(\kappa)$ is

$$\delta(\kappa) = (1-\kappa)\lambda + \sigma\kappa, \tag{7}$$

and the stationary distributions of probabilities $R_l(\kappa)$ of the service state l are defined as follows

$$R_0(\kappa) = \frac{1}{2 - B^*(\delta(\kappa))}, \quad R_1(\kappa) = \frac{1 - B^*(\delta(\kappa))}{2 - B^*(\delta(\kappa))}. \tag{8}$$

Proof. Denoting $\frac{1}{N} = \varepsilon$, in system (4) let us execute the following substitutions

$$u = \varepsilon w, \qquad H_0(u) = F_0(w,\varepsilon)\ , \qquad H_1(u,y) = F_1(w,y,\varepsilon), \tag{9}$$

then we will receive system of the equations

$$\begin{gathered}
-\lambda F_0(w,\varepsilon) + \left[\lambda e^{j\varepsilon w} - \varepsilon\sigma\right] F_1(w,\varepsilon) + e^{-j\varepsilon w}\int_0^\infty F_1(w,y,\varepsilon)\mu(y)dy \\
+j(\sigma-\lambda)\frac{\partial F_0(w,\varepsilon)}{\partial w} + j(\lambda e^{j\varepsilon w} - \sigma)\frac{\partial F_1(w,\varepsilon)}{\partial w} = 0\ , \\
\frac{\partial F_1(w,y,\varepsilon)}{\partial y} = \left[\varepsilon\sigma - \lambda - \mu(y)\right] F_1(w,y,\varepsilon) - j(\lambda-\sigma)\frac{\partial F_1(w,y,\varepsilon)}{\partial w}, \\
F_1(w,0,\varepsilon) = \lambda e^{j\varepsilon w} F_0(w,\varepsilon) + j(\lambda e^{j\varepsilon w} - \sigma)\frac{\partial F_0(w,\varepsilon)}{\partial w}.
\end{gathered} \tag{10}$$

Taking the limiting transition under conditions $\varepsilon \to 0$ and denoting $\lim\limits_{\varepsilon\to 0} F_0(w,\varepsilon) = F_0(w)$, $\lim\limits_{\varepsilon\to 0} F_1(w,y,\varepsilon) = F_1(w,y)$, then system (10) can be rewritten as

$$\begin{gathered}
\lambda\left[F_1(w) - F_0(w)\right] + \int_0^\infty F_1(w,y)\mu(y)dy \\
+j(\lambda-\sigma)\left[\frac{\partial F_1(w)}{\partial w} - \frac{\partial F_0(w)}{\partial w}\right] = 0\ , \\
\frac{\partial F_1(w,y)}{\partial y} = -\left[\lambda + \mu(y)\right] F_1(w,y) - j(\lambda-\sigma)\frac{\partial F_1(w,y)}{\partial w}, \\
F_1(w,0) = \lambda F_0(w) + j(\lambda-\sigma)\frac{\partial F_0(w)}{\partial w}.
\end{gathered} \tag{11}$$

The solution of the system (11) can be written in product-form

$$F_0(w) = R_0\Phi(w), \qquad F_1(w,y) = R_1(y)\Phi(w), \tag{12}$$

where R_0, $R_1(y)$ are the limiting probability distributions of the server state l under conditions $N \to \infty$ and $\Phi(w)$ is limiting characteristic function of the stationary distribution of random process $\frac{i(t)}{N}$. Substituting this solution into (11) we obtain

$$\begin{aligned}
&\int_0^\infty R_1(y)\mu(y)dy - (R_0 - R_1)\left[\lambda + j(\lambda-\sigma)\frac{\partial\Phi(w)/\partial w}{\Phi(w)}\right] = 0\ ,\\
&R_1^{'}(y) = -\left[\lambda + \mu(y)\right]R_1(y) - j(\lambda-\sigma)R_1(y)\frac{\partial\Phi(w)/\partial w}{\Phi(w)},\\
&R_1(0) = \lambda R_0 + j(\lambda-\sigma)R_0\frac{\partial\Phi(w)/\partial w}{\Phi(w)}.
\end{aligned} \tag{13}$$

The constant relation of the derivative function to this function allows to write down this function in the following form

$$\Phi(w) = \exp\left(jw\kappa\right), \tag{14}$$

coinciding with equality (5). Using the notation (7) the system (13) can be rewritten as

$$\begin{aligned}
&\int_0^\infty R_1(y)\mu(y)dy = \delta(\kappa)\left(R_0 - R_1\right),\\
&R_1^{'}(y) = -\left[\delta(\kappa) + \mu(y)\right]R_1(y),\\
&R_1(0) = \delta(\kappa)R_0.
\end{aligned} \tag{15}$$

From the second and third equations, taking into account the normalization condition, it is not difficult to obtain expressions for R_l of the form (8).

Let us return to system (10). Integrating the second equation of the system on y from 0 to ∞, subtracting it from the first equation, substituting the decomposition (12) and taking into account the explicit form (14) of the function $\Phi(w)$, we obtain an equation of the form

$$\int_0^\infty R_1(y)\mu(y)dy = \lambda(1-\kappa). \tag{16}$$

From (16) and the first equations of the system (15) it is obviously follows Eq. (6) for κ.

The theorem is proved.

It is interesting that Eq. (6) can have one, two or three roots $0 < \kappa < 1$. For example, for the gamma distribution function $B(x)$ with a shape parameter α and scale β with parameter values $\alpha = \beta = 2$, $\lambda = 0.29$, $\sigma = 20$, Eq. (6) has three roots, namely $\kappa_1 = 0.031$, $\kappa_2 = 0.188$ and $\kappa_3 = 0.549$.

As a rule the Eq. (6) has three roots in exceptional cases at special values of parameters and such situation arises extremely seldom. Therefore, in the following let us consider some properties of the system when the main Eq. (6) has a single root $0 < \kappa < 1$.

4 Non-ordinary Values of Throughput

First of all, let us define a measure S, called throughput, which is important for any closed retrial queuing systems.

Definition 1. *Let S be defined as the average number of customers, which are successfully served per unit time.*

Since all primary customers sooner or later will be successfully served, the throughput S naturally coincides with the intensity of the incoming (generated by the primary sources) flow. Thus, the throughput S of the considered system in the limiting condition $N \to \infty$ is equal to

$$S = \lambda(1 - \kappa). \tag{17}$$

For retrial queuing systems with collisions of customers value S can take non-ordinary values.

Let us consider the closed retrial queuing system where the service time is gamma distributed with shape parameter α and scale parameter β, with Laplace-Stieltjes transform $B^*(x)$ of the form

$$B^*(x) = \left(1 + \frac{x}{\beta}\right)^{-\alpha}. \tag{18}$$

Notice, that the average service time is α/β and in further examples we will consider a case when $\alpha = \beta$. Therefore, the average service time will be equal to unit and intensity of service as an inverse value to average time, will be also equal to unit.

Let us assume $\sigma = 20$ and the values of parameters λ and $\alpha = \beta$ indicated in the Table 1, in which the found values of throughput S of system with the collision of customers are given.

Table 1. System throughput S at various values of λ and $\alpha = \beta$

		$\alpha = \beta$							
		0.1	0.3	0.5	0.8	1	2	3	5
λ	0.5	0.494	0.486	0.478	0.458	0.432	0.099	0.024	0.0032
	1	0.968	0.909	0.827	0.633	0.478	0.089	0.023	0.0032
	5	3.911	2.441	1.503	0.748	0.487	0.084	0.022	0.0032
	10	6.101	2.961	1.628	0.760	0.488	0.083	0.022	0.0032
	15	7.441	3.173	1.672	0.764	0.488	0.083	0.022	0.0032

Table 1 demonstrates a surprising phenomenon of retrial queues with collision of customers having a throughput, which significantly greater than service intensity, and also at $\alpha > 1$ significantly less than unity.

Let us note, that throughput becomes greater than service intensity only in a case $\alpha < 1$ with increase of intensity $\lambda > 1$. For a case $\alpha > 1$, firstly, values of throughput depends weakly on value of parameter λ and, secondly, with increasing value of α it becomes close to zero.

5 Mean Sojourn Time of the Customer Under Service

For systems with collisions the sojourn time of the customer at the server has a rather complex structure, since it contains terms of zero duration when the customer from the orbit finds the server busy by another customer and a non-zero terms of the services interrupted by collisions and finally a single term of successfully completed service, after which the customer leaves the system. Let us denote by V the total mean sojourn time of the customer in the server.

Also, we will denote by $V(t)$ the residual mean sojourn time of the customer under service. For a further research we will enter the following supplementary random variable $z(t)$, equal to the residual service time, that is time interval from moment t until the end of successful service. The investigation will be carried out under limiting condition of unlimited growing number of sources, i.e. $N \to \infty$. We have the following statement

Theorem 2. *Mean sojourn time of the customer under service in a closed retrial queuing system M/GI/1//N with the collisions of customers can be defined as follows*

$$V = \frac{1 - B^*(\delta)}{\delta B^*(\delta)}, \tag{19}$$

where δ *is*

$$\delta = (1 - \kappa)\,\lambda + \sigma\kappa.$$

Proof. Let us introduce the following function of conditional mean residual sojourn time of the customer under service

$$g(z) = E\left\{V(t)|z(t) = z\right\}.$$

Using the law of total probability we obtain the following equation

$$g(z) = (1 - \delta\Delta t)(\Delta t + g(z - \Delta t)) + \delta\Delta t V + o(\Delta t). \tag{20}$$

Executing the limiting transition under conditions $\Delta t \to 0$, Eq. (20) can be rewritten as

$$g'(z) = -\delta g(z) + 1 + \delta V.$$

We obtained the Cauchy problem with the initial condition $g(0) = 0$. Let us write the solution in the form

$$g(z) = e^{-\delta z}\int\limits_0^z e^{\delta x}\left(1 + \delta V\right)dx = \left(1 + \delta V\right)\frac{1}{\delta}\left(1 - e^{-\delta z}\right). \tag{21}$$

Using the law of total probability we have

$$V = (1 - R_0)\,V + R_0 \int\limits_0^\infty g(z)dB(z).$$

Taking into account the explicit form (21) of the function $g(z)$ and performing simple transformations, we obtain

$$V = \frac{1 - B^*(\delta)}{\delta B^*(\delta)},$$

coinciding with (19).

The theorem is proved. □

We have received a formula (19) for calculating the total mean sojourn time V of the customer under service. Let us consider the influence of the system parameters on the total mean sojourn time V of the customer in the server. We will choose the same parameters of system which have been considered in the previous example when we calculating throughput S of system, namely $\sigma = 20$ and the values of parameters λ and $\alpha = \beta$ are specified in the Table 2

Table 2. Total mean sojourn time V of the customer under service at various values of λ and $\alpha = \beta$

		$\alpha = \beta$							
		0.1	0.3	0.5	0.8	1	2	3	5
λ	0.5	0.324	0.546	0.682	0.857	1	5.038	20.816	153.239
	1	0.204	0.367	0.488	0.727	1	5.576	21.695	154.766
	5	0.067	0.165	0.301	0.640	1	5.937	22.360	155.975
	10	0.046	0.140	0.280	0.632	1	5.979	22.441	156.125
	15	0.039	0.131	0.273	0.629	1	5.993	22.468	156.175

As we can see, at $\alpha < 1$ the values of total mean sojourn time V of the customer under service takes values less than unity. For $\alpha < 1$ there is a high probability of emergence of small values of service time and this fact undoubtedly influences to the total mean sojourn time V of the customer at the server and it takes rather small values. Moreover, let us remark, that with increasing values of parameter λ the values of V decrease.

In the case of $\alpha > 1$ the values of V become greater than unity. Table 2 illustrates that with increase of service parameter α the values of total mean sojourn time V of the customer under service considerably increases and reaches very large values. Let us note that in this situation parameter λ practically doesn't influence on V and with increasing parameter of service α this influence becomes less and less.

Table 2 also shows that in case of exponential service time, i.e. $\alpha = \beta = 1$, the values of total mean sojourn time V of the customer at the server is equal to unit.

Now we will return to Table 1 and we will analyze extraordinary values of throughput S of system with the help of the obtained values of total mean sojourn time V of the customer under service.

First of all, let us note, that in the case of exponential service time at any values of λ, the throughput S takes values nearly 0.5. Further, in Tables 1 and 2 it is demonstrated how much the change of total mean sojourn time V of the customer under service affects the throughput S. We can see, that with increasing the values of V, the throughput S of the system decreases and vice versa.

Previously, throughput S was defined as the average number of customers, which are successfully served per unit time. If the total service time of one customer takes a small value, then in a unit of time the server can serve several customers, hence, the system's throughput will be more unit. For example, in the case $\alpha < 0.5$ and for $\lambda > 5$ the total mean sojourn time V of the customer under service becomes small that involves increase in value of throughput S.

In the case $\alpha > 1$ Tables 1 and 2 also shows how much the big values of V affects on the values of throughput S.

6 Distribution of the Number of Retrials/Transitions of the Tagged Customer into the Orbit

Let us define the server states as follows

$$k(t) = \begin{cases} 0, & \text{server is free,} \\ 1, & \text{server is busy, but not by tagged customer,} \\ 2, & \text{server is busy by tagged customer.} \end{cases}$$

Let us denote by ν the number of transitions of the tagged customer into the orbit in the considered retrial queuing system. We should note, that random variable ν depend on N but for the simplification of notations is not shown explicitly. Applying method of the asymptotic analysis under condition of unlimited growing number of sources, we will find the probability distribution of ν. We have the following statement

Theorem 3. *Let ν be the number of transitions of the tagged customer into the orbit, then*

$$\lim_{N \to \infty} \mathsf{E}\, z^{\nu} = \frac{1-q}{1-qz}, \tag{22}$$

where value of parameter q has a form

$$q = 1 - R_0 B^*(\delta), \tag{23}$$

here

$$\delta = (1-\kappa)\,\lambda + \sigma\kappa.$$

Proof. Denote by $\nu(t)$ the residual number of transitions of the tagged customer into the orbit, that is number of transitions into the orbit of the tagged customer from moment t till the end of its successful service.

Let us introduce the conditional generating functions

$$
\begin{aligned}
G_0(z,i) &= \mathsf{E}\left\{z^{\nu(t)}|k(t)=0, i(t)=i\right\},\\
G_k(z,i,y) &= \mathsf{E}\left\{z^{\nu(t)}|k(t)=k, i(t)=i, y(t)=y\right\}, \quad k(t)=1,2.
\end{aligned}
\tag{24}
$$

Assuming that system is operating in steady state for conditional generating functions $G_0(z,i), G_k(z,i,y), k=1,2$ it is easy to obtain the following system of Kolmogorov equations

$$
\begin{aligned}
&-\left[\frac{N-i}{N}\lambda + \frac{i}{N}\sigma\right] G_0(z,i) + \frac{N-i}{N}\lambda G_1(z,i+1,0)\\
&\qquad\qquad + \frac{i-1}{N}\sigma G_1(z,i,0) + \frac{\sigma}{N} G_2(z,i,0) = 0\ ,\\
&\frac{\partial G_1(z,i,y)}{\partial y} - \left[\frac{N-i}{N}\lambda + \frac{i-1}{N}\sigma + \mu(y)\right] G_1(z,i,y)\\
&\quad + \frac{N-i}{N}\lambda G_0(z,i+1) + \frac{i-2}{N}\sigma G_0(z,i)\\
&\qquad\qquad + \frac{\sigma}{N} z G_0(z,i) + \mu(y) G_0(z,i-1) = 0\ ,\\
&\frac{\partial G_2(z,i,y)}{\partial y} - \left[\frac{N-i}{N}\lambda + \frac{i-1}{N}\sigma + \mu(y)\right] G_2(z,i,y)\\
&\qquad + \frac{N-i}{N}\lambda z G_0(z,i+1) + \frac{i-1}{N}\sigma z G_0(z,i) + \mu(y) = 0\ .
\end{aligned}
\tag{25}
$$

Denoting $\frac{1}{N} = \varepsilon$ and executing the following substitutions

$$
\begin{gathered}
i\varepsilon = x, \quad \delta(x) = (1-x)\lambda + \sigma x,\\
G_0(z,i) = F_0(z,x,\varepsilon), \quad G_k(z,i,y) = F_k(z,x,y,\varepsilon), k=1,2
\end{gathered}
\tag{26}
$$

we obtain the system of equations

$$
\begin{aligned}
&-\delta(x)F_0(z,x,\varepsilon) + (1-x)\lambda F_1(z,x+\varepsilon,0,\varepsilon)\\
&\qquad\qquad + (x-\varepsilon)\sigma F_1(z,x,0,\varepsilon) + \varepsilon\sigma F_2(z,x,0,\varepsilon) = 0,\\
&\frac{\partial F_1(z,x,y,\varepsilon)}{\partial y} - \left[\delta(x) + \mu(y) - \varepsilon\sigma\right] F_1(z,x,y,\varepsilon)\\
&\qquad + (1-x)\lambda F_0(z,x+\varepsilon,\varepsilon) + (x-2\varepsilon)\sigma F_0(z,x,\varepsilon)\\
&\qquad\qquad + \varepsilon\sigma z F_0(z,x,\varepsilon) + \mu(y) F_0(z,x-\varepsilon,\varepsilon) = 0,\\
&\frac{\partial F_2(z,x,y,\varepsilon)}{\partial y} - \left[\delta(x) + \mu(y) - \varepsilon\sigma\right] F_2(z,x,y,\varepsilon)\\
&\quad + (1-x)\lambda z F_0(z,x+\varepsilon,\varepsilon) + (x-\varepsilon)\sigma z F_0(z,x,\varepsilon) + \mu(y) = 0.
\end{aligned}
\tag{27}
$$

Carrying out the limiting transition under condition $\varepsilon \to 0$ in the system (27) and denoting $\lim\limits_{\varepsilon\to 0} F_0(z,x,\varepsilon) = F_0(z,x)$ and $\lim\limits_{\varepsilon\to 0} F_k(z,x,y,\varepsilon) = F_k(z,x,y)$, $k=1,2$, we obtain

$$\begin{aligned}
&-\delta(x)F_0(z,x) + \delta(x)F_1(z,x,0) = 0,\\
&\frac{\partial F_1(z,x,y)}{\partial y} - [\delta(x)+\mu(y)]\,(F_1(z,x,y) - F_0(z,x)) = 0,\\
&\frac{\partial F_2(z,x,y)}{\partial y} - [\delta(x)+\mu(y)]\,F_2(z,x,y) + \delta(x)zF_0(z,x)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +\mu(y) = 0.
\end{aligned} \tag{28}$$

From the first and second equations of system (28) it is easy to show that functions $F_0(z,x)$, $F_1(z,x,0)$ and $F_1(z,x,y)$ coincide, and designating by $F(z,x)$ their common value, we can write

$$F_0(z,x) = F_1(z,x,0) = F_1(z,x,y) = F(z,x).$$

Let us consider in more detail the third equation of the system (28). The solution of this equation has the form

$$\begin{aligned}
F_2(z,x,y) = e^{\int\limits_0^y [\delta(x)+\mu(v)]dv} \Bigg\{ &F_2(z,x,0)\\
&- \int\limits_0^y e^{-\int\limits_0^v [\delta(x)+\mu(u)]du} \left[\delta(x)zF(z,x) + \mu(v)\right] dv \Bigg\}.
\end{aligned} \tag{29}$$

Executing the limiting transition at $y \to \infty$ and taking into account that the first factor of the right part of equality (29) in a limiting condition tends to infinity, we can conclude that the expression in curly brackets will be equal to zero, that is

$$F_2(z,x,0) = \int\limits_0^\infty e^{-\int\limits_0^v [\delta(x)+\mu(u)]du} \left[\delta(x)zF(z,x) + \mu(v)\right] dv. \tag{30}$$

Performing simple transformations, we will receive

$$F_2(z,x,0) = \left[1 - B^*(\delta(x))\right] zF(z,x) + B^*(\delta(x)). \tag{31}$$

In order to find the function $F(z,x)$ we introduce the solution $F_0(z,x,\varepsilon)$, $F_k(z,x,y,\varepsilon)$ $k=1,2$ of system (27) in the form of the following decomposition

$$\begin{aligned}
&F_0(z,x,\varepsilon) = F(z,x) + \varepsilon f_0(z,x) + o(\varepsilon^2),\\
&F_1(z,x,y,\varepsilon) = F(z,x) + \varepsilon f_1(z,x) + o(\varepsilon^2),\\
&F_2(z,x,y,\varepsilon) = F_2(z,x,y) + \varepsilon f_2(z,x,y) + o(\varepsilon^2).
\end{aligned}$$

Substituting these decompositions into the first and second equations of system (27), we obtain equalities

$$
\begin{aligned}
&-\delta(x)\left\{F(z,x)+\varepsilon f_0(z,x)\right\}-\varepsilon\sigma F(z,x)\\
&\quad+(1-x)\lambda\left\{F(z,x)+\varepsilon\frac{\partial F(z,x)}{\partial x}+\varepsilon f_1(z,x,0)\right\}\\
&\qquad+\varepsilon\sigma F_2(z,x,0)+x\sigma\left\{F(z,x)+\varepsilon f_1(z,x,0)\right\}=o(\varepsilon^2),\\
&\varepsilon\frac{\partial f_1(z,x,y)}{\partial y}-\left[\delta(x)+\mu(y)\right]\left\{F(z,x)+\varepsilon f_1(z,x,y)\right\}+\varepsilon\sigma F(z,x)\\
&\quad+(1-x)\lambda\left\{F(z,x)+\varepsilon\frac{\partial F(z,x)}{\partial x}+\varepsilon f_0(z,x)\right\}+\varepsilon\sigma zF(z,x)\\
&\qquad+x\sigma\left\{F(z,x)+\varepsilon f_0(z,x)\right\}-2\varepsilon\sigma F(z,x)\\
&\qquad\quad+\mu(y)\left\{F(z,x)-\varepsilon\tfrac{\partial F(z,x)}{\partial x}+\varepsilon f_0(z,x)\right\}=o(\varepsilon^2).
\end{aligned}
$$

Equating here coefficients at identical degrees ε, for functions $f_0(z,x)$, $f_1(z,x,y)$ and $f_2(z,x,y)$ we obtain following system

$$
\begin{aligned}
&-\delta(x)f_0(z,x)+\delta(x)f_1(z,x,0)\\
&\qquad=\sigma\left\{F(z,x)-F_2(z,x,0)\right\}-(1-x)\lambda\frac{\partial F(z,x)}{\partial x},\\
&-\left[\delta(x)+\mu(y)\right]f_1(z,x,y)+\left[\delta(x)+\mu(y)\right]f_0(z,x)+\frac{\partial f_1(z,x,y)}{\partial y}\\
&\qquad=(1-z)\sigma F(z,x)+\left[\mu(y)-(1-x)\lambda\right]\frac{\partial F(z,x)}{\partial x}.
\end{aligned}
\tag{32}
$$

Multiplying the first equality on $R_0(x)$, second on $R_1(x,y)$, adding this products and integrating the received equality on y from 0 to ∞ we obtain

$$
\begin{aligned}
&\left\{\delta(x)\left[R_1(x)-R_0(x)\right]+\int_0^\infty\mu(y)R_1(x,y)dy\right\}f_0(z,x)\\
&\quad+\delta(x)R_0(x)f_1(z,x,0)-\int_0^\infty\left[\delta(x)+\mu(y)\right]f_1(z,x,y)R_1(x,y)dy\\
&\quad+\int_0^\infty\frac{\partial f_1(z,x,y)}{\partial y}R_1(x,y)dy=(1-z)\sigma F(z,x)R_1(x)\\
&\qquad+\sigma R_0(x)\left[F(z,x)-F_2(z,x,0)\right]\\
&\qquad\quad+\left[\int_0^\infty\mu(y)R_1(x,y)dy-(1-x)\lambda\right]\frac{\partial F(z,x)}{\partial x}.
\end{aligned}
\tag{33}
$$

Substituting into (33) $x=\kappa$ let us denote $R_0(x)=R_0$, $R_1(x,y)=R_1(y)$, $F_0(z,x)=F_0(z)$, $F_2(z,x,y)=F_2(z,y)$. Taking into account equalities (15) from Theorem 1, it is not difficult to obtain that the left part of equality (33) is equal to zero. Owing to (16), we obtain

$$
(1-z)\sigma F(z)+\sigma R_0\left[zF(z)-F_2(z,0)\right]=0,
$$

which, taking into account equality (31), can be rewritten as

$$\Big\{ 1 - z\,[1 - R_0 B^*(\delta)] \Big\}\, F(z) = R_0 B^*(\delta). \tag{34}$$

Denote

$$q = 1 - R_0 B^*(\delta),$$

coinciding with (23), we have received that function $F(z)$ has the form

$$F(z) = \frac{1-q}{1-qz},$$

which coincide with (22).

The theorem is proved.

From the proved theorem it is obviously follows that the probability distribution $P\{\nu = n\}$, $n = \overline{0,\infty}$ of the number of transitions of the tagged customer into the orbit is geometric and has the form

$$P\{\nu = n\} = (1-q)q^n, \quad n = \overline{0,\infty}. \tag{35}$$

Let us consider the influence of the system parameters on the values of mean number of retrials ν. We will choose the same parameters of system which have been considered in the previous examples, namely $\sigma = 20$ and the values of parameters λ and $\alpha = \beta$ are specified in the Table 3.

Table 3. Mean number of retrials for various values of λ and $\alpha = \beta$

		$\alpha = \beta$							
		0.1	0.3	0.5	0.8	1	2	3	5
λ	0.5	0.470	1.131	1.86	3.65	6.32	162.7	793	6091
	1	0.656	2.002	4.19	11.57	21.83	204.1	848	6172
	5	1.114	4.192	9.31	22.73	37.07	234.4	891	6236
	10	1.278	4.755	10.28	24.30	39.02	238.1	896	6244
	15	1.355	4.969	10.63	24.83	39.67	239.3	898	6247

From Table 3 it is shown that the mean number of retrials obviously depends on service parameter α that is the expected and logical result. It should be noted that for small values of parameter α influence of parameter λ on mean number of retrials is significant and obvious. But with the increase of parameter α this influence decreases and already for great values of α practically disappears.

The following conclusions can be drawn from the Tables 1, 2 and 3. The presence of extraordinary throughput values S, values of mean number of retrials and mean sojourn time of the customer under service is a consequence of the collision of customers and the admissibility of repeated attempts of service the same customer. Duration of the customer service for repeated attempts has the same probability distribution $B(x)$, but its repeated realization, naturally, accepts various values. If for the distribution $B(x)$ there is a high probability of emergence of small values of service time as in the gamma distribution with the shape parameter $\alpha < 1$, then a small number of retries is sufficient to realize a small value of the service time which will be successful and, as shown in the Table 1, the throughput will be greater than intensity of service.

If the small values of the service time are unlikely for the probability distribution $B(x)$, as in the gamma distribution with the shape parameter $\alpha > 1$, then the number of unsuccessful attempts of service becomes big, as we can see in the Table 3, the server works without results, the mean sojourn time of the customer under service is increase (Table 2) and the throughput S becomes close to zero.

Let us denote by W the sojourn/waiting time of the tagged customer in the orbit. On the basis of the Theorem 3 we can formulate the following statement

Theorem 4. *Characteristic function of the sojourn/waiting time W of the tagged customer in an orbit has the form*

$$\mathsf{E}e^{juW} = (1-q) + q\frac{\sigma(1-q)}{\sigma(1-q) - juN}. \tag{36}$$

Proof. The proof is trivial.

7 Conclusions

In this paper, a finite-source retrial queuing system with collisions of customers was considered. It was shown that at some combinations of system parameter values the throughput takes on exotic values. In addition, in the present paper the sojourn time analysis of the considered system was presented. The research has been conducted by method of asymptotic analysis under condition that number of sources tends to infinity while the primary request generation rate, retrial rate tend to zero. As the result of the investigation it was shown that probability distribution of the number of retrials/transitions of the customer into the orbit is geometric with given parameters, and the normalized sojourn time of the customer in the orbit has Generalized Exponential distribution. The mean sojourn time of the customer under service was obtained. Examples and tables demonstrated the novelty of the investigations.

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