CUSTOMER MOTION IN QUEUEING MODELS: THE USE OF TANGENT VECTOR FIELDS

Dieter Baum [1] and János Sztrik [2]

[1] University of Trier, Dept. IV, D–54286 Trier, GERMANY
e-mail: `baum@uni-trier.de`

[2] Institute of Mathematics and Informatics,
University of Debrecen, Debrecen, P.O.Box 12, H–4010, HUNGARY
e-mail: `jsztrik@math.klte.hu`

Abstract: Recently, no-waiting stations with spatial arrival process and arbitrary service time distribution have been analyzed [2, 3]. Such stations represent adequate models of, for example, mobile communication networks based on code division multiple access (CDMA) techniques. Also, the inclusion of customer movements in space in such models has been performed. In [4, 5], movements were represented as group operations, which map a region $\mathfrak{R}$ of interest into itself. In this paper we present a method to construct these group operations by means of velocity tangent vector fields. Field curves are to be interpreted as curves along which customers move. We show, how the varying behaviour of customers in space (e.g. calls in progress in a mobile communication network model) in a system fed by a spatial arrival process can be adequately described. Former results allow to compute the transient as well as equilibrium distribution of numbers of customers in any Borel subset of $\mathfrak{R}$.

1. Introduction

The inclusion of spatial aspects into batch Markovian arrival processes (BMAPs) *over time* has been set going in 1997 by a proposal of Latouche and Ramaswami [1]. This approach, however, was restricted to a special construction, creating spatial point patterns with respect to arrival times. A different and more general approach has been suggested in 1998 by Baum and Kalashnikov [2, 3]. Here the rate matrices of a BMAP have reference to Borel subsets S of some region $\Re$ in a complete separable metric space. The method also applies to other stochastic processes, e.g. to Markov additive processes of arrivals (MA-PAs)[1], and has been successfully employed in [4, 5] to analyze queueing models with moving customers. To the author's knowledge, with this exception the concept of motion in models of communication systems has so far not been included in analyses, rather has been considered independently of temporal and spatial model characteristics. In the work of Baccelli, Klein, Lebourges, and Zuyev [6, 7] the selection of initial customer locations was performed by "static" samplings of point patterns from point processes, whereas the temporal characteristics have been included afterwards via the marking of random fields with velocity distribution parameters. Compared with these results our approach directly combines service activity and spatial arrival pattern with customer motion in space, offering higher versatility for the modelling of systems with temporal and spatial dynamics.

In this paper we concretize the construction of mappings which reflect customer motion by introducing velocity tangent vector fields and interpreting the field curves as those along which customers move.
The paper is organized as follows. In section 2 we summarize some definitions and terminology in connection with spatial arrival processes and corresponding queueing models. In section 3 we construct group operations with means of tangent vector fields, and demonstrate how the diversity of possible customer motions can be reflected by a superposition of these fields. Section 4 contains the analysis of no-waiting models with spatial arrival process, general service time distribution, and moving customers. Subsections 4.1 and 4.2 are devoted to the $SBMAP/G/\infty$ queue and its finite capacity variant, respectively. The treatment of customer motion along field curves of overlapping vector fields is presented in subsection 4.3. In section 5 we give a short résumé.

---

[1]The acronyms MAP and BMAP have been introduced for "Markovian arrival processes", and "batch Markovian arrival processes", respectively, which form subclasses of the class of Markov additive processes of arrivals. Therefore, we use MaP as a label for "Markov additive processes", and MAPA for "Markov additive processes of arrivals".

2. Terminology and Background

2.1. Terminology

Let J be a Markov process over $[0, \infty)$ with discrete state space $E = \{1, \ldots, m\}$, and let the pair $(\mathbf{X}, J) = \{(\mathbf{X}_t, J_t) : t \in \mathfrak{T}\}$ describe a time-homogeneous Markov-additive process (MaP) on the state space $\mathbb{N}_0^d \times E$. $\mathbf{X}$ is called the additive component, and J the Markov component of the MaP $(\mathbf{X}, J)$ [8, 9, 10]. If $\mathbf{X}$ takes values only in the set of nonnegative d-dimensional integer vectors we speak of a *multivariate Markov additive process of arrivals* or a MAPA for short. A univariate MAPA ($d = 1$) is a BMAP as introduced by Neuts [11] and Lucantoni [12]. For $d > 1$ we speak of a *multivariate* MAPA.

Random vectors are denoted by bold face u.c. Roman letters ($\mathbf{X}, \mathbf{Y}$ *etc.*), elements of $\mathbb{R}^d$ are denoted by bold face l.c. Roman letters: $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, and matrices by u.c. Roman letters (A, B etc.). Relations $\leq, \geq, <, >, \ldots$ on $\mathbb{R}^d$ are to be understood to hold for each component, i.e. $\mathbf{a} \leq \mathbf{b}$ with $\mathbf{a} \neq \mathbf{b}$ means, that any component of $\mathbf{a}$ is smaller or equal to the corresponding component of $\mathbf{b}$, whereas at least one component of $\mathbf{b}$ is greater than its counterpart in $\mathbf{a}$. $\delta_{\mathbf{x}}$ shall denote the Kronecker function, which equals 1, if $x \geq 0$, and 0 otherwise.

For sequences $\mathfrak{A} = \{A_0, A_1, \ldots\}$, $\mathfrak{B} = \{B_0, B_1, \ldots\}$ of $(m \times m)$-matrices a discrete convolution $\mathfrak{A} * \mathfrak{B}$ is defined by $(\mathfrak{A} * \mathfrak{B})_v = \sum_{\ell=0}^{v} A_\ell \cdot B_{v-\ell}$. The unit element in the semi-group of such sequences of $(m \times m)$-matrices with respect to the operator "$*$" is the sequence $\mathbf{1} = \{\mathbf{I}, \mathbf{O}, \mathbf{O}, \ldots\}$, where $\mathbf{I}$ and $\mathbf{O}$ are the unit matrix and the null matrix, respectively. The convolutional exponential for $\mathfrak{A} \cdot t = \{A_0 \cdot t, A_1 \cdot t, \ldots\}$ is represented by $e^{*\mathfrak{A} \cdot t} := \sum_{\nu=0}^{\infty} (t^\nu / \nu!) \cdot \mathfrak{A}^{*\nu}$ (t a scalar).

The MaP state transition probability measure reads $P_{A;ij}(t) = \mathbb{P}\{\mathbf{X}_t \in A, J_t = j \mid \mathbf{X}_0 = \mathbf{0}, J_0 = i\}$ for $A \in \mathcal{B}^d$, where $\mathcal{B}^d$ is the σ-algebra of Borel subsets of $\mathbb{R}^d$, and $P_{\mathbf{n};ij}(t) = \mathbb{P}\{\mathbf{X}_t = \mathbf{n}, J_t = j \mid \mathbf{X}_0 = \mathbf{0}, J_0 = i\}$ in case of a MAPA. For $i, j \in E$ and $\mathbf{k}, \mathbf{n} \in \mathbb{N}_0^d$ the value $D_{\mathbf{n};ij}$ describes the transition rate from $(\mathbf{k}, i)$ to $(\mathbf{k} + \mathbf{n}, j)$. Corresponding matrices are denoted accordingly, e.g. $P_{\mathbf{n}}(t) = ((P_{\mathbf{n};ij}(t)))_{i,j \in \{1,\ldots,m\}}$, and $D_{\mathbf{n}} = ((D_{\mathbf{n};ij}))_{i,j \in \{1,\ldots,m\}}$. In order to serialize sets of elements (matrices) with vectorial index we use a bijection $g : \mathbb{N}_0^d \to \mathbb{N}_0$ (for a detailed descripten see [2, 13]), and write $\Delta = \{D_{g^{-1}(0)}, D_{g^{-1}(1)}, \ldots\}$, and $\Pi(t) = \{P_{g^{-1}(0)}(t), P_{g^{-1}(1)}(t), \ldots\}$, respectively. A MAPA is called stable [10], if $\lambda_i = \sum_{(j,\mathbf{n}) \neq (i,\mathbf{0})} D_{\mathbf{n};ij} < \infty$ for all $i \in E$. We assume throughout that this stability condition holds. The Chapman-Kolmogorov (C-K) equations as well as the C-K differential equations for a stable MAPA may be written as $\Pi(s+t) = \Pi(s) * \Pi(t)$, $\frac{d}{dt}\Pi(t) = \Delta * \Pi(t)$, respectively. The solution of the C-K-differential equation is obtained as $\Pi(t) = e^{*\Delta t}$ (cf. [3]). For a univariate MAPA, i.e. a BMAP, we clearly have the corresponding expressions with scalars instead of vectors. Its rate matrices are D_n

($n \in \mathbb{N}_0$), with the generator of the phase process given by $D = \sum_{n=0}^{\infty} D_n$. The equilibrium customer state vector of any $BMAP/G/*$ - station, if it exists independently of the start phase i, is denoted by $\vec{\mathbf{y}} = (\mathbf{y}_0, \mathbf{y}_1, \ldots)$, where each $\mathbf{y}_r$ is an m-vector $\mathbf{y}_r = (y_{r,1}, y_{r,2}, \ldots, y_{r,m})$ with $y_{r,j} = \lim_{t \to \infty} \mathbb{P}\{N_t = r, J_t = j \mid N_0 = 0, J_0 = i\}$, N_t the r.v. of the number of customers in the system at time t ($r \in \mathbb{N}_0$). Accordingly, for a multivariate MAPA, we use bold face subscripts in $\vec{\mathbf{y}} = (\mathbf{y_0}, \mathbf{y_1}, \ldots)$ and $\mathbf{y_r} = (y_{\mathbf{r},1}, y_{\mathbf{r},2}, \ldots, y_{\mathbf{r},m})$, where now $y_{\mathbf{r},j} = \lim_{t \to \infty} \mathbb{P}\{\mathbf{N}_t = \mathbf{r}, J_t = j \mid \mathbf{N}_0 = \mathbf{0}, J_0 = i\}$, $\mathbf{N}_t \in \mathbb{N}_0^d$ the random vector of class specific customer numbers in the system at time t.

2.2. Spatial Arrival Processes

In its most general form the spatial version of a BMAP has been given by Breuer [14], who admitted the phase space to be uncountable (continuous); this version is beyond our scope. For finite phase space E the general definition has been formulated by Baum and Kalashnikov [3]. A rough characterization of an SMAPA may be seen in the concept of a common MAPA, whose rate matrices are equipped with probability measures over Borel subsets of some Polish space (usually the $\mathbb{R}^2$). More precisely, set $p_i(\mathbf{0}, i) = -1$ for all $i \in E$, and let $p_i(\mathbf{n}, j)$ for $(\mathbf{n}, j) \neq (\mathbf{0}, i)$ be the probability for a phase transition from i to j together with an $\mathbf{n}$-arrival. Let $\mathfrak{R}$ be a finite subset of $\mathbb{R}^2$ or, more general, a two-dimensional manifold in $\mathbb{R}^3$, $\mathcal{B}(\mathfrak{R})$ the σ-algebra of Borel subsets of $\mathfrak{R}$. Further, given a MAPA with rate matrices $D_{\mathbf{n}} = ((D_{\mathbf{n};ij}))_{i,j \in \{1,\ldots,m\}}$, $D_{\mathbf{n};ij} = \lambda_i \cdot p_i(\mathbf{n}, j)$, let $\Phi = \{\phi_{ij;\mathbf{n}} : i, j \in E, \mathbf{n} \in \mathbb{N}_0^d\}$ be a family of probability measures over $\mathcal{B}(\mathfrak{R})$. Then the spatial MAPA (SMAPA) is defined by its S-specific rate matrices for any $S \in \mathcal{B}(\mathfrak{R})$:

$$
\begin{aligned}
D_{\mathbf{n};ij}(S) &= \lambda_i \cdot p_i(\mathbf{n}, j) \cdot \phi_{ij;\mathbf{n}}(S) \quad \text{for } \mathbf{n} \neq \mathbf{0}, \\
D_{\mathbf{0};ij}(S) &= \lambda_i \cdot p_i(\mathbf{0}, j) + \sum_{\mathbf{n} \neq \mathbf{0}} p_i(\mathbf{n}, j) \cdot \phi_{ij;\mathbf{n}}(\mathfrak{R} \setminus S).
\end{aligned}
\tag{2.1}
$$

The transition probabilities of the SMAPA read $P_{\mathbf{n};ij}(S, t) = \mathbb{P}\{\mathbf{X}_t(S) = \mathbf{n}, J_t = j \mid \mathbf{X}_0(S) = \mathbf{0}, J_0 = i\}$. The additive component $\mathbf{X}(S)$ "counts" multivariate batch arrivals in subset $S \in \mathcal{B}(\mathfrak{R})$. Due to the Markov property the sequence $\Pi(S, t) = \{P(S, t)_{g^{-1}(0)}, P(S, t)_{g^{-1}(1)}, P(S, t)_{g^{-1}(2)}, \ldots\}$ is given by $\Pi(S, t) = e^{*\Delta(S)t}$ with $\Delta(S) = \{D(S)_{g^{-1}(0)}, D(S)_{g^{-1}(1)}, D(S)_{g^{-1}(2)}, \ldots\}$, $S \in \mathcal{B}(\mathfrak{R})$, $g : \mathbb{N}_0^d \to \mathbb{N}_0$ a bijection. The joint distribution for any family $\mathbf{S} = \{S_1, S_2, \ldots, S_\kappa\}$ of subsets $S_i \in \mathcal{B}(\mathfrak{R})$ can be easily represented in convolutional exponential form, as has been shown in [3].

Another variant of the arrival processes is that of level dependent MA-PAs. The class specific numbers k_c of customers in a system at time epoch t determine what is called the *level vector* $\mathbf{k} = (k_1, k_2, \ldots, k_d)$, or the *level* for short, of the system. The investigation of level dependent BMAPs has been

performed by Hofmann [15, 16]; he showed, that there are similar expressions for the transition matrices as in the case of level independent rate matrices. The definition of a level dependent version of a MAPA is straight forward. On condition that any level is associated with the same phase process J_t, we write $D_{\mathbf{n}}^{(\mathbf{k})}$ for the rate matrix of vectorial batch arrivals of type $\mathbf{n} = (n_1, n_2, \ldots, n_d)$ at a time epoch when the level is $\mathbf{k}$.

2.3. Matrix Differential Equations Associated with $SBMAP/G/\infty$ - and $SMAPA/G/\mathbf{c}/\mathbf{c}$ - Stations

For no-waiting models of type $SMAPA/G/\infty$ and $SMAPA/G/\mathbf{c}/\mathbf{c}$ the transient as well as steady state probability distributions have been obtained in [2] and [4] as solutions of matrix differential equations. We first consider an SBMAP as arrival process to an infinite server system. Let $N_{u,t}(S)$ be the number of customers who arrived in $S \in \mathcal{B}(\mathfrak{R})$ until time u and are still in service at time $t \geq u$, and set $Q_{r;ij}(S; u, t) = \mathbb{P}(N_{u,t}(S) = r, J_u = j \mid J_0 = i)$, $Q_r(S; u, t) = (Q_{r;ij}(S; u, t))_{i,j\in E}$, and $\mathcal{Q}(S; u, t) = \{Q_0(S; u, t), Q_1(S; u, t), \ldots\}$. Further, write $b_k(n, F(\xi))$ for the Bernoulli probability $\binom{n}{k}(1 - F(\xi))^k F^{n-k}(\xi)$, where $F(\xi)$ is the cdf of service time, and set $R_k(S; \xi) = \sum_{n=k}^{\infty} D_n(S)b_k(n, F(\xi))$, and $\mathcal{R}(S; \xi) = \{R_0 k(S; \xi), R_1(S; \xi), \ldots\}$. Then the following matrix differential equation holds, which can be solved by iteration [2]:

$$\frac{\partial}{\partial u} Q_r(S; u, t) = \sum_{\ell=0}^{r} Q_\ell(S; u, t) R_{r-\ell}(S; t - u) =: (\mathcal{Q}(S; u, t) * \mathcal{R}(S; t - u))_r .$$

$$(2.2)$$

In case of an $SMAPA/G/\mathbf{c}/\mathbf{c}$ system (a loss system), the arrival intensity depends upon the numbers of customers being in service. Let $\mathbf{l} = (l_1, \ldots, l_d)$ denote a vector of resident customers; we call it the actual *level*. Then $Q_{\mathbf{r}}^{(\mathbf{l})}(S; u, t)$ is the matrix of phase depending probabilities for the fact, that at time u the level is $\mathbf{l}$, and a vector $\mathbf{r} \leq \mathbf{l}$ of customers is observed in S, who stay in S until time t. Using a similar approach as for the $SBMAP/G/\infty$ system, the corresponding matrix differential equation for the loss system has been formulated in [5] in the following way: Let

$$U_{\mathbf{l}-\mathbf{k},\mathbf{r}-\mathbf{m}}^{(\mathbf{k})}(S; u, t) \; := \; D_{\mathbf{l}-\mathbf{k}}^{(\mathbf{k})}(S) \times \prod_{c=1}^{d} b_{r_c - m_c}(l_c - k_c; F_c(t - u)),$$

$$V_{\mathbf{l}-\mathbf{k}}^{(\mathbf{k})}(S; u, t) \; := \; D_{\mathbf{l}-\mathbf{k}}^{(\mathbf{k})}(S) \times \prod_{c=1}^{d} b_0(l_c - k_c; F_c(t - u)) + D_{\mathbf{l}-\mathbf{k}}^{(\mathbf{k})}(\mathfrak{R} \setminus S),$$

$$W_{c;\,\mathbf{r}}^{(\mathbf{l},\mathbf{e}_c)}(S; u, t) \; := \; Q_{\mathbf{r}}^{(\mathbf{l}+\mathbf{e}_c)}(S; u, t) \cdot (l_c + 1 - r_c) \cdot \delta_{\mathbf{c}-\mathbf{l}-\mathbf{e}_c}$$

$$- \; Q_{\mathbf{r}}^{(\mathbf{l})}(S; u, t) \cdot (l_c - r_c) ; \qquad (2.3)$$

then

$$\frac{\partial Q_{\mathbf{r}}^{(\mathbf{l})}(S;\, u, t)}{\partial u} = \sum_{\substack{0 \leq \mathbf{m} \leq \mathbf{r} \\ \mathbf{m} \neq \mathbf{r}}} \sum_{\mathbf{k}=\mathbf{m}}^{\mathbf{l}-(\mathbf{r}-\mathbf{m})} Q_{\mathbf{m}}^{(\mathbf{k})}(S;\, u, t) \cdot U_{\mathbf{l}-\mathbf{k},\, \mathbf{r}-\mathbf{m}}^{(\mathbf{k})}(S;\, u, t) \; +$$

$$+ \; Q_{\mathbf{r}}^{(\mathbf{l})}(S;\; u, t) \cdot D_{\mathbf{0}}^{(\mathbf{l})}(S) + \sum_{\substack{\mathbf{r} \leq \mathbf{k} \leq \mathbf{l} \\ \mathbf{k} \neq \mathbf{l}}} Q_{\mathbf{r}}^{(\mathbf{k})}(S;\, u, t) \cdot V_{\mathbf{l}-\mathbf{k}}^{(\mathbf{k})}(S;\, u, t) \; +$$

$$+ \; \sum_{c=1}^{d} h_c(u) \cdot W_{c;\, \mathbf{r}}^{(\mathbf{l}, \mathbf{e}_c)}(S\,;\, u, t)\,. \qquad (2.4)$$

This equation again can be solved by an iteration scheme [13, 5].

3. Motion in Space

Consider the region $\mathfrak{R}$ as some finite part of the earth's surface in a landscape. Assume that $\mathfrak{R}$ is smooth in the sense that $\mathfrak{R}$ represents a two-dimensional and (at least twofold) differentiable manifold, which, in fact, is embedded here in the $\mathbb{R}^3$. This means that $\mathfrak{R}$ is a class-k surface in $\mathbb{R}^3$ in the sense, that for each point p of $\mathfrak{R}$ there exists a proper patch[2] in $\mathfrak{R}$ whose image contains a neighbourhood of p in $\mathfrak{R}$ ($k \geq 2$). A well known condition for class-k surfaces in $\mathbb{R}^3$ says that if $\mathbf{x} : G \to \mathbb{R}^3$ and $\mathbf{y} : H \to \mathbb{R}^3$ are patches whose images overlap, then the composite functions $\mathbf{x}^{-1} \circ \mathbf{y}$ and $\mathbf{y}^{-1} \circ \mathbf{x}$ are $k-$fold differentiable mappings on open sets of $\mathbb{R}^2$.

Since with our applications in mind it will be sufficient for all practical cases to consider the suface $\mathfrak{R}$ as such a one which is covered already by the image of one patch $\mathbf{x} : G \to \mathbb{R}^3$ alone, we may assume w.l.o.g. that the local Cartesian coordinates u^1, u^2 in G determine valid coordinates u^1, u^2 for all points of $\mathfrak{R}$. Each point p of $\mathfrak{R}$ then has a representation as a single-valued real vector function $\mathbf{x} = \mathbf{x}(u^1, u^2)$, and the Jacobian determinant $\det((\frac{\partial x_i}{\partial u^j}))_{i \in \{1,2,3\}, j \in \{1,2\}}$ for $\mathbf{x} = (x_1, x_2, x_3)$ has rank 2.

3.1. Tangent Vector Fields

Let $\mathfrak{O}$ denote a simply connected subregion of $\mathfrak{R}$, and assume that there is a family of curves $\kappa_c : \Gamma \to \mathfrak{O}$ which covers $\mathfrak{O}$, defined as the family of field curves of a tangent vector field $\mathbf{V}$ over $\mathfrak{O}$ (here Γ is an open interval in $\mathbb{R}^1$, and each c is a parameter)[3]. $\mathbf{V}$, in our application, can be interpreted as a field

---

[2]A patch (also coordinate patch or chart) $\mathbf{x} : G \to \mathbb{R}^3$ is a one-to-one regular mapping of an open set G of $\mathbb{R}^2$ into $\mathbb{R}^3$. A mapping $\mathbb{R}^2 \to \mathbb{R}^3$ is called regular, if its Jacobian matrix has rank 2.

[3]The set of curves $\{\kappa_c : c \in I = (a, b) \subset \mathbb{R}^1\}$ depends continuously on the parameter c.

of velocity tangent vectors, the curves being seen as the traces along which customers move, the length of each vector giving the velocity.

In general, if a curve $\kappa : \Gamma \to \mathfrak{D}$ lies in the image $\mathbf{x}(G)$ of a patch $\mathbf{x}$, then there exist unique differentiable functions u^1, u^2 on Γ, such that $\mathbf{x}(u^1(\gamma), u^2(\gamma)) = \kappa(\gamma)$ for all $\gamma \in \Gamma$. Thus, with $I = [a, b) \subset \mathbb{R}^1$, the family $\{\kappa_c : c \in I\}$ is a set of curves $\{\mathbf{x}(\gamma, c) : c \in I\}$, each of which is associated with a parameter subinterval Γ_c of Γ, where $\Gamma_c = \{\gamma : \alpha_c \le \gamma < \beta_c\} \subset \Gamma$ depends on $\mathfrak{D}$. The tangent vector field $\mathbf{V}$ is given by the set $\{\frac{\partial}{\partial \gamma}\mathbf{x}(\gamma, c) = \mathbf{x}_{u^1} \cdot \dot{u}^1(\gamma, c) + \mathbf{x}_{u^2} \cdot \dot{u}^2(\gamma, c) : \gamma \in \Gamma_c, c \in I \subset \mathbb{R}^1\}^4$, which corresponds to the solutions of a set of differential equations

$$\frac{d}{dt}\mathbf{x}(\gamma) = \mathbf{f}(\mathbf{x}(\gamma)) = (f_1(\mathbf{x}(\gamma)), f_2(\mathbf{x}(\gamma)), f_3(\mathbf{x}(\gamma))) \tag{3.1}$$

together with a family of initial conditions

$$\mathbf{x}(\gamma)|_{\gamma=\gamma_0(c)} = \mathbf{x}_0(c). \tag{3.2}$$

In order to guarantee that for any point $\mathbf{z} \in \mathfrak{D}$ one and only one curve κ_c passes through $\mathbf{z}$ it is necessary and sufficient, that there are $c \in I$, $\gamma_0(c) \in \Gamma_c$, and a solution $\mathbf{x}(\gamma) = \mathbf{x}(u^1(\gamma), u^2(\gamma))$ with $\mathbf{x}(\gamma_0(c)) = \mathbf{x}(u^1(\gamma_0, c), u^2(\gamma_0, c)) = \mathbf{z}$, and the functions $f_i(x_1, x_2, x_3)$ are continuous with continuous partial derivatives with respect to x_j within a vicinity of $\gamma = \gamma_0(c), x_j = x_{0j}$ $(i, j \in \{1, 2, 3\}$, $(x_{01}, x_{02}, x_{03}) = \mathbf{x}_0(c))$. These conditions are satisfied per definition for a vector field.

The tangent vector field $\mathbf{V}$ over $\mathfrak{D}$ allows to define a family of endomorphisms on $\mathfrak{D}$. Let $\tau \in \mathbb{R}_+$ be fixed, $\Gamma_c = \{\gamma : \alpha_c \le \gamma < \beta_c\}$. For each field curve $\kappa_c = \{\mathbf{x}(\gamma, c) : \gamma \in \Gamma_c\}$ in $\mathfrak{D}$ define a mapping $\Psi_\tau(c) : \kappa_c \to \kappa_c$ by $\Psi_\tau(c)(\mathbf{x}(\gamma, c)) = \mathbf{x}(\gamma + \tau, c)$, if $\gamma + \tau \in \Gamma_c$, and $\Psi_\tau(c)(\mathbf{x}(\gamma, c)) = \mathbf{x}(\alpha_c + \gamma + \tau - \beta_c, c)$, if $\gamma + \tau \ge \beta_c$ (cyclic shift). Since $\{\kappa_c : c \in I\}$ covers $\mathfrak{D}$ with one and only one curve passing through a point of $\mathfrak{D}$, the family of mappings $\{\Psi_\tau(c) : c \in I\}$ defines an endomorphism Υ_τ on $\mathfrak{D}$ by $\Upsilon_\tau(\mathbf{z}) = \Psi_\tau(c)(\mathbf{z})$ for $\mathbf{z} = \mathbf{x}(\gamma, c)$. We introduce the following notation:

$$\Upsilon_{-\tau}[S] = \{\mathbf{x}(u^1, u^2) \in \mathfrak{D} : \Upsilon_\tau(\mathbf{x}(u^1, u^2)) \in S\} = \tilde{S}_\tau$$

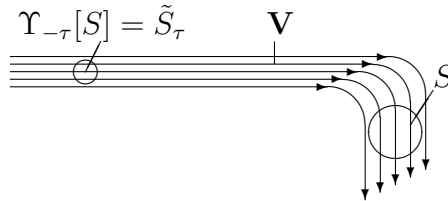for each measurable subset S of $\mathfrak{D}$, and $\tau \in \mathbb{R}_+$ (see Fig. 1 below).

Figure 1

$^4\mathbf{x}_{u^i} = \frac{\partial}{\partial u^i}\mathbf{x}, \; \dot{u}^i = \frac{d}{d\gamma}u^i, \; i = 1, 2.$

In applications there may be several streams of moving customers, and movements occur in different directions (Fig. 2). In order to include these aspects, it is necessary to consider a family $\mathcal{V} = \{\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_W\}$ of tangent vector fields together with the corresponding family of open sets $\{\mathfrak{O}_1, \mathfrak{O}_2, \ldots, \mathfrak{O}_W\}$ of $\mathfrak{R}$. Let

$$\tilde{S}_\tau^{(h)} = \Upsilon_{-\tau}^{(h)}[S] = \{\mathbf{x}(u^1, u^2) \in \mathfrak{O}_h : \Upsilon_\tau^{(h)}(\mathbf{x}(u^1, u^2)) \in S\},$$

where $\Upsilon_\tau^{(h)}$ is the endomorphism on $\mathfrak{O}_h$ defined through the tangent vector field $\mathbf{V}_h$. Then

$$\tilde{S}_\tau = \bigcup_{h=1}^{W} \tilde{S}_\tau^{(h)} = \bigcup_{h=1}^{W} \Upsilon_{-\tau}^{(h)}[S]$$

for each $\tau \leq t$. We include in this setting the possibility that, for some h, $\Upsilon_\tau^{(h)}$ is the identity mapping on $\mathfrak{O}_h$. The combination of all $\Upsilon_\tau^{(h)} : \mathfrak{O}_h \to \mathfrak{O}_h$ does not, in general, define a mapping Υ over $\bigcup_{h=1}^{W} \mathfrak{O}_h$. It is this fact that causes us to write $\tilde{S}_\tau$ instead of $\Upsilon_{-\tau}[S]$ in the general case.
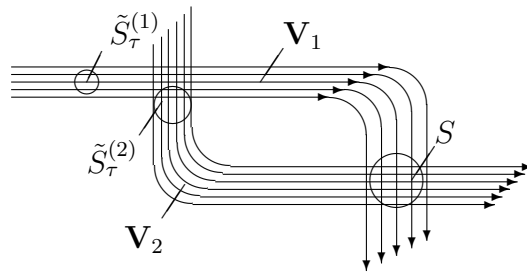

Figure 2

4. No-Waiting Models

In this section we show, that the matrix differential equations (2.2), (2.4) for $SBMAP/G/\infty$ and $SMAPA/G/\mathbf{c}/\mathbf{c}$ models hold without major changes for the case, that customers move in space. This has been previously demonstrated for non-specified group mappings of type Υ_s in [5].

4.1. $SBMAP/G/\infty$ with Customer Motion

Consider a measurable subset of $\bigcup_{h=1}^{W} \mathfrak{O}_h \subset \mathfrak{R}$, each $\mathfrak{O}_h$ being covered by the family $\{\kappa_c^{(h)} : c \in I^{(h)}\}$ of field curves of a velocity tangent field as described in the previous section. Using otherwise a similar notation as in section 2.3, we replace the probability matrices $Q_r(S; u, t)$ as follows: Let $\tau := t - u$ for $u \leq t$, and let $\tilde{Q}_{r;ij}(S; u, t)$ denote the probability that at time u the phase is j, and the number of those customers in $\tilde{S}_\tau$, who will be resident in S at time

t is r, given that at time zero the system was empty, and the phase was i. Let N_t be the total number of customers in the system at time t. Then, with the notation explained in section 2.3,

$$\tilde{Q}_{r;ij}(S; u, t) = \mathbb{P}\{N_{u,t}(\tilde{S}_\tau) = r, J_u = j \mid N_t = 0, J_0 = i\}.$$

As usual, $\tilde{Q}_r(S; u, t)$ denotes the matrix formed by the entries $\tilde{Q}_{r;ij}(S; u, t)$ $(i, j \in E)$. Since the mappings Υ_s are continuous, we have

$$\lim_{d\tau \to 0} \{\tilde{S}_\tau \setminus \tilde{S}_{\tau-d\tau}\} = \emptyset, \quad \forall \tau > 0.$$

Denote $d\tilde{S} = \tilde{S}_\tau \setminus \tilde{S}_{\tau-d\tau}$, and let us assume, that each probability measure $\phi_{ij;n}$ over $\mathcal{B}(\mathfrak{R})$ from the family $\Phi = \{\phi_{ij;n} : i, j \in E, n \in \mathbb{N}_0\}$ satisfies

$$\Phi_{ij;n}(d\tilde{S}) = O(d\tau) \, \forall i, j \in E \ n \geq 0. \tag{4.1}$$

To describe the dynamics of the process, we observe the number $X(\tilde{S}_s, d\tau)$ of arrivals in the varying set $\tilde{S}_s$ during an infinitesimal interval of length $d\tau$, i.e. for $\tau - d\tau \leq s \leq \tau$. Let $X(\tilde{S}_s, d\tau) = n$ with probability $\tilde{P}_{n;ij}(\tilde{S}_s, d\tau, t)$ (i the starting phase, j the destination phase), and set

$$\tilde{P}_n(\tilde{S}_s, d\tau, t) = (\tilde{P}_{n;ij}(\tilde{S}_s, d\tau, t))_{i,j \in E}.$$

Lemma 1. *For small $d\tau$, $\tilde{P}_n(\tilde{S}_s, d\tau, t)$ equals $P_n(\tilde{S}_{\tau-d\tau}; d\tau)$ up to quantities of order $o(d\tau)$, where $P_n(\tilde{S}_{\tau-d\tau}; d\tau) = \mathbb{P}\{\mathbf{X}_{d\tau}(\tilde{S}_{\tau-d\tau}) = n, J_{d\tau} = j \mid \mathbf{X}_0(\tilde{S}_{\tau-d\tau}) = 0, J_0 = i\}$.*

Proof. Obviously,

$$\begin{aligned}
\delta_{n0}\mathbf{I} \ + \ & D_n(\tilde{S}_\tau \cap \tilde{S}_{\tau-d\tau}) \, d\tau + o(d\tau) \ \leq \ \tilde{P}_n(\tilde{S}_s, d\tau, t) \\
& \leq \ \delta_{n0}\mathbf{I} + D_n(\tilde{S}_\tau \cup \tilde{S}_{\tau-d\tau}) \, d\tau + o(d\tau).
\end{aligned}$$

By definition of D_n and according to (4.1), we have $D_{n;ij}(\tilde{S}_\tau \cap \tilde{S}_{\tau-d\tau}) = \lambda_i p_i(n, j)\Phi_{ij;n}(\tilde{S}_\tau) + O(d\tau)$, and $D_{n;ij}(\tilde{S}_\tau \cup \tilde{S}_{\tau-d\tau}) = \lambda_i p_i(n, j)\Phi_{ij;n}(\tilde{S}_\tau) + O(d\tau)$. Therefore, $\tilde{P}_n(\tilde{S}_s, d\tau, t) = \delta_{n0}\mathbf{I} + D_n(\tilde{S}_\tau) \, d\tau + o(d\tau)$, which completes the proof. $\square$

Theorem 1. *The matrices $\tilde{Q}_r(S; t-\tau, t)$ satisfy the following differential equation:*

$$\frac{\partial}{\partial u}\tilde{Q}_r(S; u, t) = (\tilde{\mathcal{Q}}(S; u, t) * \tilde{\mathcal{R}}(S; t - u))_r, \tag{4.2}$$

where $u = t - \tau$, and

$$\tilde{R}_\ell(S; t - u) = \sum_{n=\ell}^{\infty} D_n(\tilde{S}_\tau) \cdot b_\ell(n; F(\tau)),$$

$$\tilde{\mathcal{Q}}(S; u, t) = \{\tilde{Q}_0(S; u, t), \tilde{Q}_1(S; u, t), \tilde{Q}_2(S; u, t), \ldots\},$$

$$\tilde{\mathcal{R}}(S; t - u) = \{\tilde{R}_0(S; t - u), \tilde{R}_1(S; t - u), \tilde{R}_2(S; t - u), \ldots\}.$$

Proof. $\tilde{Q}_r(S; t - \tau + d\tau, t)$ can be expressed for $\tau - d\tau \leq s \leq \tau$ as

$$\tilde{Q}_r(S; t - \tau + d\tau, t) = \sum_{k=0}^{r} \sum_{n=r-k}^{\infty} \tilde{Q}_r(S; t - \tau, t) \cdot \tilde{P}_n(\tilde{S}_s; d\tau, t) \cdot b_{r-k}(n, F(\tau)).$$

Exploiting the relationship $\tilde{P}_n(\tilde{S}_s, d\tau, t) = \delta_{n0} \mathbf{I} + D_n(\tilde{S}_\tau) \, d\tau + o(d\tau)$, we obtain

$$
\begin{aligned}
\tilde{Q}_r(S; t - \tau + d\tau, t) &= \sum_{k=0}^{r} \sum_{n=r-k}^{\infty} \tilde{Q}_r(S; t - \tau, t) \cdot D_n(\tilde{S}_\tau; d\tau) \cdot b_{r-k}(n, F(\tau)) \\
&\quad + \tilde{Q}_r(S; t - \tau, t) + M \cdot o(d\tau),
\end{aligned}
$$

with some finite constant matrix M, such that

$$\frac{\partial}{\partial \tau} \tilde{Q}_r(S; t - \tau, t) = \sum_{k=0}^{r} \sum_{n=r-k}^{\infty} \tilde{Q}_r(S; t - \tau, t) \cdot D_n(\tilde{S}_\tau) \cdot b_{r-k}(n, F(\tau)).$$

$(r = 0, 1, \ldots)$. This equation is equivalent to (4.2). □

Equation (4.2) completely corresponds to equation (2.2) in section 2.3 and, therefore, its solution is the same as presented in [2], theorem 1, and [5]. To be more precise, we are again led to the following expression for the transient state probabilities:

$$\tilde{Q}_r(S; t) = \int_{u=0}^{t} d\tilde{Q}_r(S; u, t). \tag{4.3}$$

The sequence of matrices $\tilde{Q}(S; t) = \{\tilde{Q}_0(S; t), \tilde{Q}_1(S; t), \tilde{Q}_2(S; t), \ldots\}$ can be computed iteratively. Notice, that $\mathbf{1} = \{\mathbf{I}, \mathbf{O}, \mathbf{O}, \ldots\}$; then

$$
\begin{aligned}
\tilde{\mathcal{Q}}^{(0)}(S; u) &= \mathbf{1} \\
\tilde{\mathcal{Q}}^{(i+1)}(S; u) &= \mathbf{1} + \int_0^u \tilde{\mathcal{Q}}^{(i)}(S; \xi) * \tilde{\mathcal{R}}(S; t - \xi) d\xi \quad \text{for } i \in \mathbb{N}_0 \tag{4.4}
\end{aligned}
$$

The iteration converges uniquely (see e.g. [17] for the matrix analogon), i.e.

$$\tilde{\mathcal{Q}}(S; t) = \lim_{i \to \infty} \tilde{\mathcal{Q}}^{(i)}(S; u) \Big|_{u=t}. \tag{4.5}$$

The equilibrium distribution exists for each stable BMAP and service time distribution with finite mean, and is given as the sequence of m-vectors $\vec{\mathbf{y}}(S) = \{\mathbf{y}_0(S), \mathbf{y}_1(S), \ldots\}$, where $\mathbf{y}_r(S) = \{y_{r;1}(S), y_{r;2}(S), \ldots, y_{r;m}(S)\}$ and

$$y_{r;j}(S) = \mathbb{P}\{N(S) = r, J = j\} = \lim_{t \to \infty} \tilde{Q}_{r;ij}(S; t) \; \forall i \in E. \tag{4.6}$$

4.2. $SMAPA/G/\mathbf{c}/\mathbf{c}$ with Customer Motion

Let $\mathbf{c} = (c_1, \ldots, c_d)$ denote a capacity vector. Following the same line of reasoning as in the previous section the analysis of an $SMAPA/G/\mathbf{c}/\mathbf{c}$ system with moving customers can be performed by just replacing the rate matrices $D_{\mathbf{l-k}}^{(\mathbf{k})}(S)$ in equation (2.4) by the rate matrices associated with subregion $\tilde{S}_\tau$, where $\tilde{S}_\tau = \bigcup_{h=1}^{W} \tilde{S}_\tau^{(h)}$, and

$$\tilde{S}_\tau^{(h)} := \Upsilon_{-\tau}^{(h)}[S] = \{\mathbf{x}(u^1, u^2) \in \mathfrak{O}_h : \Upsilon_\tau^{(h)}(\mathbf{x}(u^1, u^2)) \in S\}.$$

We refer to the definition of the shortcuts $U_{\mathbf{l-k},\mathbf{r-m}}^{(\mathbf{k})}(S; u, t)$ and $V_{\mathbf{l-k}}^{(\mathbf{k})}(S; u, t)$ given in section 2.3, in which the set S has to be replaced by the domain set $\tilde{S}_\tau = \tilde{S}_{t-u}$:

$$U_{\mathbf{l-k},\mathbf{r-m}}^{(\mathbf{k})}(\tilde{S}_{t-u}; u, t) := D_{\mathbf{l-k}}^{(\mathbf{k})}(\tilde{S}_{t-u}) \cdot \prod_{c=1}^{d} b_{r_c-m_c}(l_c - k_c; F_c(t-u)),$$

$$V_{\mathbf{l-k}}^{(\mathbf{k})}(\tilde{S}_{t-u}; u, t) := D_{\mathbf{l-k}}^{(\mathbf{k})}(\tilde{S}_{t-u}) \cdot \prod_{c=1}^{d} b_0(l_c - k_c; F_c(t-u))$$
$$+ \quad D_{\mathbf{l-k}}^{(\mathbf{k})}(\mathfrak{R} \setminus \tilde{S}_{t-u}),$$

Then the matrix differential equation for the case of moving customers takes the following form:

$$\frac{\partial Q_{\mathbf{r}}^{(\mathbf{l})}(S; u, t)}{\partial u} = \sum_{\substack{0 \leq \mathbf{m} \leq \mathbf{r} \\ \mathbf{m} \neq \mathbf{r}}} \sum_{\mathbf{k=m}}^{\mathbf{l-(r-m)}} Q_{\mathbf{m}}^{(\mathbf{k})}(S; u, t) \cdot U_{\mathbf{l-k,\,r-m}}^{(\mathbf{k})}(\tilde{S}_{t-u}; u, t) \quad +$$

$$+ \quad Q_{\mathbf{r}}^{(\mathbf{l})}(S; u, t) \cdot D_{\mathbf{0}}^{(\mathbf{l})}(\tilde{S}_{t-u}) + \sum_{\substack{\mathbf{r} \leq \mathbf{k} \leq \mathbf{l} \\ \mathbf{k} \neq \mathbf{l}}} Q_{\mathbf{r}}^{(\mathbf{k})}(S; u, t) \cdot V_{\mathbf{l-k}}^{(\mathbf{k})}(\tilde{S}_{t-u}; u, t) \quad +$$

$$+ \quad \sum_{c=1}^{d} h_c(u) \cdot W_{c;\,\mathbf{r}}^{(\mathbf{l}, \mathbf{e}_c)}(S; u, t) \qquad\qquad (4.7)$$

(see [5]). To solve (4.7), we return to the results of [5]. We first state, that the sum

$$Q_{\mathbf{r}}(S; u, t) = \sum_{\mathbf{l} \geq \mathbf{r}} Q_{\mathbf{r}}^{(\mathbf{l})}(S; u, t)$$

completely describes the observed state at time $u \leq t$ with respect to the number of t-resident customers in S. The transient state probability matrices for epoch t can be obtained as $Q_{\mathbf{r}}(S; t) = Q_{\mathbf{r}}(S; t, t)$. Since the model

under consideration is stable, these matrices tend to finite limits as $t \to \infty$, thereby providing the sequence of equilibrium state probability vectors with matrix components $\mathcal{Q}_{\mathbf{r}}(S) = (Q_{\mathbf{r};1}(S), Q_{\mathbf{r};2}(S), \ldots, Q_{\mathbf{r};m}(S))$, where $Q_{\mathbf{r};j}(S) = \lim_{t \to \infty} Q_{\mathbf{r};ij}(S;t)$, $i \in E = \{1, \ldots, m\}$, $\mathbf{0} \leq \mathbf{r} \leq \mathbf{c}$. The remaining task, therefore, is to find a solution to (4.7). This has been performed in [13] for the non-spatial case by transforming the equation into a homogeneous matrix-vector differential equation: The doubly indexed (matrix) structures are ordered into single-indexed (vector) structures (we speak of vectors and matrices, although sequences and block matrices of $(m \times m)$-matrices are meant; this is justified by the fact, that the analysis of corresponding expressions is the same as in case of "normal" vectors and matrices with scalar entries). Let $\beta : \mathbb{N}_0{}^d \times \mathbb{N}_0{}^d \to \mathbb{N}_0$ denote a bijection, that uniquely maps a pair $(\mathbf{l}, \mathbf{r}) \in \{(\mathbf{x}, \mathbf{y}) : \mathbf{0} \leq \mathbf{x}, \mathbf{y} \leq \mathbf{c}\}$ of vector indices to an integer $\beta(\mathbf{l}, \mathbf{r})$. Then, by means of β, we order the set of matrices in sequence (vector) form:

$$Q_{\mathbf{r}}^{(\mathbf{l})}(S; u, t) =: Q^{[\beta(\mathbf{l}, \mathbf{r})]}(S; u, t), \quad \mathbf{0} \leq \mathbf{r} \leq \mathbf{l} \leq \mathbf{c},$$

where $Q_{\mathbf{m}}^{(\mathbf{n})}(S; u, t) = \mathbf{O}$, if $\mathbf{m} \not\leq \mathbf{n}$ or $\mathbf{n} \not\leq \mathbf{c}$. Let $K = \prod_{i=1}^{d}(1 + c_i)$ denote the total number of possible system levels, and set $a_k = K^2 - 1$. We thus define the sequence

$$\mathfrak{Q}^{[\beta]}(S; u, t) = (Q^{[0]}(S; u, t), Q^{[1]}(S; u, t), \ldots, Q^{[a_K]}(S; u, t)).$$

The following theorem is then an immediate consequence from (4.7).

Theorem 2. *For $\mathbf{r} \leq \mathbf{l} \leq \mathbf{c}$ and $u \leq t$, the equation (4.7) takes the form*

$$\frac{\partial \mathfrak{Q}^{[\beta]}(S; u, t)}{\partial u} = \mathfrak{Q}^{[\beta]}(S; u, t) \mathcal{H}_{\mathbf{c}}(S; u, t), \tag{4.8}$$

where $\mathcal{H}_{\mathbf{c}}(S; u, t)$ is a $(a_K \times a_K)$-matrix of $(m \times m)$-matrices defined as follows. Assume, that $\beta^{-1}(i) = (\mathbf{m}, \mathbf{k})$ and $\beta^{-1}(j) = (\mathbf{r}, \mathbf{l})$. Then the (i, j)-entry in $\mathcal{H}_{\mathbf{c}}(S; u, t)$ (denoted below for simplicity as $(\mathcal{H})_{i,j}$) is represented by the following expression:

$$(\mathcal{H})_{i,j} = \begin{cases} \mathbf{O} & \text{if } \mathbf{m} \not\leq \mathbf{k}, \text{ or } \mathbf{r} \not\leq \mathbf{l}, \text{ or } \mathbf{m} \not\leq \mathbf{r}, \\ U_{\mathbf{l}-\mathbf{k},\mathbf{r}-\mathbf{m}}^{(\mathbf{k})}(\tilde{S}_{t-u}; u, t) & \text{if } \mathbf{m} < \mathbf{r}, \mathbf{m} \leq \mathbf{k} \leq \mathbf{l} - (\mathbf{r} - \mathbf{m}), \\ V_{\mathbf{l}-\mathbf{k}}^{(\mathbf{k})}(\tilde{S}_{t-u}; u, t) & \text{if } \mathbf{m} = \mathbf{r}, \mathbf{r} \leq \mathbf{k}, \mathbf{k} \not\geq \mathbf{l}, \\ D_{\mathbf{0}}^{(\mathbf{l})}(\tilde{S}_{t-u}) - \mathbf{I} \cdot \sum_{c=1}^{d} h_c(u)(l_c - r_c) & \text{if } \mathbf{m} = \mathbf{r} \text{ and } \mathbf{k} = \mathbf{l}, \\ \mathbf{I} \cdot \sum_{c=1}^{d} h_c(u)(l_c - r_c + 1) & \text{if } \mathbf{m} = \mathbf{r} \text{ and } \mathbf{k} = \mathbf{l} + \mathbf{e}_c, \\ & \text{and } \mathbf{c} - (\mathbf{l} + \mathbf{e}_c) \geq \mathbf{0}, \\ \mathbf{O} & \text{in all other cases.} \end{cases} \tag{4.9}$$

By analogy from matrix analysis it can be shown, that the solution to (4.8) is obtained by first solving the matrix-matrix differential equation

$$\frac{\partial \mathcal{X}(S;\, u, t)}{\partial u} = \mathcal{X}(S;\, u, t) \cdot \mathcal{H}_{\mathbf{c}}(S;\, u, t)\,, \tag{4.10}$$

where $\mathcal{X}$ is an $(a_K \times a_K)$-block matrix with $\mathcal{X}(S; 0, t) = \mathbf{I}$, and then setting

$$\mathfrak{Q}^{[\beta]}(S;\, u, t) = \mathfrak{Q}^{[\beta]}(S; 0, t)\, \mathcal{X}(S;\, u, t)\,,$$

(see [5], and compare the corresponding analysis in [17]). (4.10) is solved by iteration:

$$
\begin{aligned}
\mathcal{X}_0(S;\, u, t) &= \mathbf{I}, \\
\mathcal{X}_{i+1}(S;\, u, t) &= \mathbf{I} + \int_{s=0}^{u} \mathcal{X}_i(S; s, t)\mathcal{H}_{\mathbf{c}}(S;\, s; t)\, ds \quad i \geq 0\,.
\end{aligned} \tag{4.11}
$$

The existence and uniqueness of this limit follows again from analogy to the case of matrices with scalar entries [17]. When $\mathfrak{Q}^{[\beta]}(S;\, u, t)$ has been computed for $u = t$, one can find the transient state probability matrices with elements $Q_{\mathbf{r}; i, j}(S; t) = \mathbb{P}\{\mathbf{N}_t(S) = \mathbf{r}, J_t = j \mid \mathbf{N}_0(S) = \mathbf{0}, J_0 = i\}$ using the equality $Q_{\mathbf{r}}(S; t) = \sum_{\mathbf{l} \geq \mathbf{r}} Q^{[\beta(\mathbf{r}, \mathbf{l})]}(S;\, t, t) = \sum_{\mathbf{l} \geq \mathbf{r}} Q_{\mathbf{r}}^{(\mathbf{l})}(S;\, t, t)$. The equilibrium state probabilities are obtained as the limits $Q_{\mathbf{r}; j}(S) = \lim_{t \to \infty} Q_{\mathbf{r}; ij}(S; t)$. They exist, independently of $i \in E$, due to the stability of the model.

4.3. Stream Specific Characteristics

Considering the equilibrium distribution $Q_{\mathbf{r}; j}(S; t) = \mathbb{P}\{\mathbf{N}(S) = \mathbf{r}, J_t = j\}$ one may also ask for the distribution of stream specific customer numbers in a subset S. This problem can be solved by assigning non-negative weights ω_h to the vector fields $\mathbf{V}_h$, $h \in \{1, \ldots, W\}$, and determining, for each $\tau \leq t$, the share of customers belonging to a field $\mathbf{V}_h$ in the domain set $\tilde{S}_\tau$. Let us introduce some notation: A family $\{\mathfrak{D}_{i_\nu} : \nu = 1, 2, \ldots, k\} \subset \{\mathfrak{D}_1, \mathfrak{D}_2, \ldots, \mathfrak{D}_W\}$ with the properties

$$
\begin{aligned}
\tilde{S} \cap \mathfrak{D}_{i_\nu} &\neq \emptyset \quad \text{for } 1 \leq \nu \leq k, \tag{4.12} \\
\tilde{S} \cap \mathfrak{D}_\ell &= \emptyset, \quad \ell \notin \{i_1, \ldots, i_k\}, \tag{4.13}
\end{aligned}
$$

for $\tilde{S} \in \mathcal{B}(\mathfrak{R})$ is called an $\tilde{S}$ *intersecting family*. Obviously, such a family is uniquely determined by $\tilde{S}$ and the set of support sets $\mathfrak{D}_i$, $i = 1, \ldots, W$. Recall, that for each measurable set $S \in \mathcal{B}(\mathfrak{R})$ we have

$$\tilde{S}_\tau^{(h)} := \Upsilon_{-\tau}^{(h)}[S] = \{\mathbf{x}(u^1, u^2) \in \mathfrak{D}_h : \Upsilon_\tau^{(h)}(\mathbf{x}(u^1, u^2)) \in S\},$$

where $\Upsilon_\tau^{(h)}$ is the endomorphism on $\mathfrak{D}_h$ defined through the tangent vector field $\mathbf{V}_h$. Let us denote by $\mathcal{F}_\tau$ the set of all those subsets $\{i_1, \ldots, i_k\} \subset \{1, \ldots, W\}$,

which determine an $\tilde{S}_\tau^{(h)}$ intersecting family for any $h \in \{1, \ldots, W\}$. For fixed h let $\{\mathfrak{D}_{i_1}, \mathfrak{D}_{i_2}, \ldots, \mathfrak{D}_{i_k}\}$ denote the $\tilde{S}_\tau^{(h)}$ intersecting family, and set $\{i_1, i_2, \ldots, i_k\} \setminus \{h\} =: \mathcal{I}^{(h)}$. Define

$$\Sigma_h^{(h)}(\tau) := \tilde{S}_\tau^{(h)} \setminus \Big(\bigcup_{i_\nu \in \mathcal{I}^{(h)}} \mathfrak{D}_{i_\nu} \Big),$$

and for each subset $\{j_1, j_2, \ldots, j_\ell\} \subset \mathcal{I}^{(h)}$ consider the point sets

$$\Sigma_{j_1, j_2, \ldots, j_\ell}^{(h)}(\tau) := \tilde{S}_\tau^{(h)} \cap \Big(\bigcap_{\mu=1}^\ell \mathfrak{D}_{j_\mu} \Big) \setminus \Big(\bigcup_{i_\nu \notin \{j_1, j_2, \ldots, j_\ell\}} \mathfrak{D}_{i_\nu} \Big),$$

such that

$$\Sigma_h^{(h)}(\tau) \cup \Big(\bigcup_{\{j_1, j_2, \ldots, j_\ell\} \subset \mathcal{I}^{(h)}} \Sigma_{j_1, j_2, \ldots, j_\ell}^{(h)}(\tau) \Big) = \tilde{S}_\tau^{(h)},$$

and

$$\Sigma_h^{(h)}(\tau) \cap \Big(\bigcup_{\{j_1, j_2, \ldots, j_\ell\} \subset \mathcal{I}^{(h)}} \Sigma_{j_1, j_2, \ldots, j_\ell}^{(h)}(\tau) \Big) = \emptyset.$$

In this way we form a partition of each set $\tilde{S}_\tau^{(h)}$ by the subsets $\Sigma_h^{(h)}$ and $\Sigma_{j_1, \ldots, j_\ell}^{(h)}$, $\{j_1, \ldots, j_\ell\} \subset \mathcal{I}^{(h)}$, $h \in \{1, \ldots, W\}$.
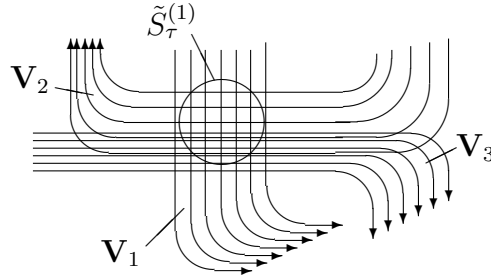

Figure 3

As an example, in Fig. 3 we have, for $h = 1$, $\Sigma_1^{(1)}(\tau) = \tilde{S}_\tau^{(1)} \setminus \Big(\mathfrak{D}_2 \cup \mathfrak{D}_3 \Big)$, $\Sigma_2^{(1)}(\tau) = \tilde{S}_\tau^{(1)} \cap \mathfrak{D}_2 \setminus \mathfrak{D}_3$, $\Sigma_3^{(1)}(\tau) = \tilde{S}_\tau^{(1)} \cap \mathfrak{D}_3 \setminus \mathfrak{D}_2$, $\Sigma_{2,3}^{(1)}(\tau) = \tilde{S}_\tau^{(1)} \cap \Big(\mathfrak{D}_2 \cap \mathfrak{D}_3 \Big)$.

Notice, that for any $h \in \{1, \ldots, W\}$, $\{j_1, \ldots, j_\ell\} \subset \mathcal{I}^{(h)}$, and $j_n \in \{j_1, \ldots, j_\ell\}$ we have

$$\Sigma_{j_1, \ldots, j_\ell}^{(h)}(\tau) = \Sigma_{j_1, \ldots, j_n, h, j_{n+1}, \ldots, j_\ell}^{(j_n)}(\tau).$$

Accordingly, we may define

$$\Sigma_{\nu_1, \ldots, \nu_n}(\tau) = \begin{cases} \emptyset & \text{if } \{\nu_1, \ldots, \nu_\ell\} \notin \mathcal{F}_\tau \\ \Sigma_{\nu_2, \ldots, \nu_n}^{(\nu_1)}(\tau) & \text{if } \{\nu_1, \ldots, \nu_\ell\} \in \mathcal{F}_\tau \end{cases},$$

such that $\left\{\Sigma_{\nu_1,\ldots,\nu_n}(\tau) : \{\nu_1,\ldots,\nu_n\} \subset \{1,\ldots,W\}\right\}$ represents the family of all partition sets related to the collection of all $\tilde{S}_\tau^{(h)}$, $h \in \{1,\ldots,W\}$. This means, that for any $\tau \leq t$ we have constructed a set of disjoint subsets $\Sigma_{\nu_1,\ldots,\nu_n}(\tau)$ with the property, that

$$\bigcup_{h\in\{1,\ldots,W\}} \bigcup_{\{\nu_1,\ldots,\nu_n\}\in\mathcal{F}_\tau} \Sigma_{\nu_1,\ldots,\nu_n}(\tau) = \bigcup_{h\in\{1,\ldots,W\}} \tilde{S}_\tau^{(h)}.$$

Then the measure $\phi_{ij;n}(\tilde{S}_\tau)$ can be written in the form

$$\phi_{ij;n}(\tilde{S}_\tau) = \sum_{h\in\{1,\ldots,W\}} \sum_{\{\nu_1,\ldots,\nu_n\}\in\mathcal{F}_\tau} \phi_{ij;n}(\Sigma_{\nu_1,\ldots,\nu_n}(\tau)).$$

The solution to our problem now consists in redefining the measures $\phi_{ij;n}(\tilde{S}_\tau)$ for each $h \in \{1,\ldots,W\}$ by assigning to each $\phi_{ij;n}(\Sigma_{\nu_1,\ldots,\nu_n}(\tau))$ the weight

$$\Omega_h = \begin{cases} \omega_h/\sum_{j=1}^n \omega_{\nu_j} & \text{if } h \in \{\nu_1,\ldots,\nu_n\} \\ 0 & \text{if } h \notin \{\nu_1,\ldots,\nu_n\} \end{cases},$$

i.e. setting

$$\phi_{ij;n}^{(h)}(\tilde{S}_\tau) = \sum_{h\in\{1,\ldots,W\}} \sum_{\{\nu_1,\ldots,\nu_n\}\in\mathcal{F}_\tau} \phi_{ij;n}(\Sigma_{\nu_1,\ldots,\nu_n}(\tau)) \cdot \Omega_h. \qquad (4.14)$$

The solution of the differential equation corresponding to (4.7), where the spatial rate matrices are redefined with means of (4.14), yields the stream specific information which was required.

5. Summary and Comments

Based on the construction of velocity tangent vector fields, we have shown in this paper, how spatial models with moving customers can be analyzed, providing transient as well as steady state distributions of number of customers in any subset of the spatial area of interest. $SMAPA/G/\infty$ and $SMAPA/G/\mathbf{c}/\mathbf{c}$ models with moving customers play an important role for the analysis of mobile communication networks, in particular of CDMA based cell networks. Considering one cell in such a network (corresponding to the area $\mathfrak{R}$ above), it may be of importance to know the distribution of customers in certain subareas as, for instance, blocks of buildings or streets or highways. In such an environment the above mentioned fields of velocity tangent vectors will be defined according to the movement of mobiles along the streets. The velocities have to be estimated as averages, and the different streams (vector fields) have to mirror main streams of automobile motion. The analysis of a corresponding

$SMAPA/G/\mathbf{c}/\mathbf{c}$ model can help then to check the validity of the cell covering structure, and to determine possible bottlenecks. During a radio network definition phase, the performance analysis can be used to determine the location of demand nodes, characterizing clusters of average call request locations.

In view of the complexity of all deductions in this paper it is clear, that for practical use these theoretical results have to be supplemented by the development of software packages. This is not only necessary for solving the matrix differential equations, but also for preparing numerical results in a form that an engineer can understand and use.

Acknowledgement

References

[1] Latouche, G., Ramaswami, V., *Spatial Point Patterns of Phase Type.* Proc. 15th Int. Teletraffic Congress ITC 15, Washington D.C., Volume 2a, p. 381-390, 1997.

[2] Baum, D., *The Infinite Server Queue with Markov Additive Arrivals in Space.* Proc. of the RareEvents'99 Int. Conf., ed. by V. V. Kalashnikov and A. M. Andronov, Riga Aviation University, Riga, Latvia, 1999. Also: Research Rep. No. 98-31, University of Trier, 1998

[3] Baum, D., Kalashnikov, V., *Spatial Generalization of BMAPs with Finite Phase Space.* Research Rep. No. 98-11, University of Trier, 1998. To appear in the Journal of Mathematical Sciences

[4] Baum, D., Kalashnikov, V., *No-Waiting stations with Moving Customers.* Abstract Proceedings of the XXIth Intern. Seminar on Stability Problems for Stochastic Models, Eger, Hungary, Jan. 2001

[5] Baum, D., Kalashnikov, V., *Stochastic Models for Communication Networks with Moving Customers.* Information Processes, Vol. 1, No. 1, pp 1-18, 2001

[6] Baccelli, F., Klein, M., Lebourges, M., and Zuyev, S., *Stochastic Geometry and Architecture of Communication Networks.* J. Telecom. Systems, 1996. Also in: Selected Proceedings of the Third INFOCOM Telecommunications Conference, 1995

[7] Baccelli, F., Zuyev, S., *Stochastic Geometry Models of Mobile Communication Networks.* In: Frontiers in Queueing, Probability and Stochastics Series, ed. by Jewgeni H. Dshalalow. CRC Press, Boca Raton, 1997

[8] Çinlar, E., *Markov additive processes I, II*, Zeitschrift f. Wahrscheinlichkeitstheorie und verwandte Gebiete **24**, 1972

[9] Prabhu, N. U., *Markov renewal and Markov-additive processes - A review and some new results.* In: *Proc. of KAIST Math. Workshop* **6**, ed. by B. D. Choi and J. W. Yi, Korea Adv. Inst. of Science and Tech., Taejon, 1991

[10] Pacheco, A., Prabhu, N. U., *Markov-Additive Processes of Arrivals.* In: *Advances in Queueing: Theory, Methods, and Open Problems*, ed. by Jewgeni H. Dshalalow, CRC Press, Boca Raton, 1995

[11] Neuts, M. F., *A Versatile Markovian Point Process.* J. Appl. Prob., 16: 764-779, 1979

[12] Lucantoni, D. M., *New results on the single server queue with a batch Markovian arrival process.* Communications in Statistics: Stochastic Models, 7(1), p.1-46, Marcel Dekker Inc., 1991

[13] Baum, D., *Multi Server Queues with Markov Additive Arrivals.* Proc. 3rd Int. Conf. on Matrix Analytic Methods (MAM3), Leuven, Belgium, 2000. Also in *Advances in Algorithmic Methods for Stochastic Models* (eds. G. Latouche, P. Taylor), Notable Publications, New Jersey, 2000

[14] Breuer, L., *Spatial Queues with Infinitely Many Servers.* Research Rep. No. 99-04, University of Trier, 1999.

[15] Hofmann, J., *The BMAP/G/1 Queue with Level Dependent Arrivals.* Doctoral thesis, University of Trier, February 1999.

[16] Hofmann, J., *The BMAP/G/1 Queue with Level Dependent Arrivals — An Overview.* Select Proceedings of the 4th INFORMS Telecom. Conf., Boca Raton, 1998. To appear in: Special Issue of Telecommunication Systems, 1999

[17] Bellman, R., *Introduction to Matrix Analysis.* SIAM Society for Industrial and Applied Mathematics, Philadelphia, 1995