Comparative Analysis of Methods of Residual and Elapsed Service Time in the Study of the Closed Retrial Queuing System M/GI/1//N with Collision of the Customers and Unreliable Server

Anatoly Nazarov[1,2], Janos Sztrik[3(✉)], and Anna Kvach[1]

[1] National Research Tomsk State University, 36 Lenina Avenue, Tomsk 634050, Russia
kvach_as@mail.ru, nazarov.tsu@gmail.com

[2] Department of Applied Probability and Informatics, Peoples' Friendship University of Russia, Miklukho-Maklaya Street 6, Moscow 117198, Russia

[3] University of Debrecen, Debrecen, Hungary
sztrik.janos@inf.unideb.hu

Abstract. The aim of the present paper is to investigate a finite-source M/GI/1 retrial queuing system with collision of the customers where the server is subject to random breakdowns and repairs depending on whether it is idle or busy. The method of elapsed service time and the method of residual service time are considered using asymptotic approach under the condition of unlimited growing number of sources. It is proved, as it was expected, that basic characteristics of the system, such as the stationary probability distribution of the server states and the asymptotic average of the normalized number of customers in the system are the same and do not depend on the applied method.

Keywords: Finite-source queuing system · Closed queuing systems · Retrial queue · Collision · Server breakdowns and repairs · Unreliable server · Asymptotic analysis · Method of residual service time · Method of elapsed service time

1 Introduction

Retrial queues have been widely used to model many problems arising in telephone switching systems, telecommunication networks, computer networks and computer systems, call centers, wireless communication systems, etc.

In many practical situations it is important to take into account the fact that the rate of generation of new primary calls decreases as the number of customers in the system increases. This can be done with the help of finite-source, or quasi-random input models. Moreover, usually in the study of various

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queuing systems, servers are assumed to be absolutely reliable. But in practice it is necessary to take into account the possibility of failure and repair of the server. Finite-source retrial queues with unreliable server have been investigated in, for example [1,2,9,10]. Recent results on retrial queues with collisions can be found in, for example [3,6,8].

The aim of the present paper is to investigate such systems which has the above mention properties, that is finite-source, retrial, collision, and non-reliability of the server. The introduced model is a generalization of the systems treated in [4,5,7]. Two methods are considered using asymptotic approach under the condition of unlimited growing number of sources. It is proved, as it was expected, that basic characteristics of the system, such as the stationary probability distribution of the server states and the asymptotic average of the normalized number of customers in the system are the same and do not depend on the applied method.

The rest of the paper is organized as follows. In Sect. 2 the description of the model is given, the corresponding two-dimensional non-Markov process is defined. In Sects. 3 and 4 the residual service time method and the elapsed service time method are considered by using asymptotic analysis, respectively. Section 5 is devoted to the comparison of the offered methods. Finally, the paper ends with a Conclusion.

2 Model Description and Notations

Let us consider a closed retrial queuing system of type M/GI/1//N with collision of the customers and unreliable server (Fig. 1). The number of sources is N and each of them can generate a primary request during an exponentially distributed time with rate λ/N. A source cannot generate a new call until end of the successful service of this customer. If a primary customer finds the server idle, he enters into service immediately, in which the required service time has

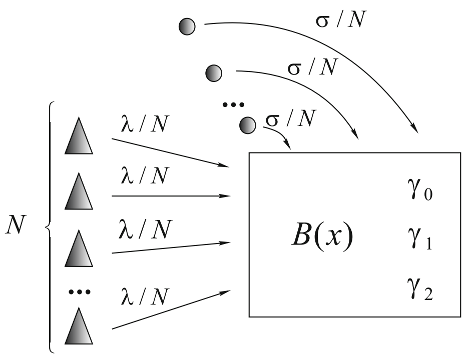

Fig. 1. Closed retrial queuing system M/GI/1//N with collision of the customers and unreliable server

a probability distribution function $B(x)$. Let us denote its hazard rate function by $\mu(y) = B^{'}(y)(1 - B(y))^{-1}$ and Laplace -Stieltjes transform by $B^{*}(y)$, respectively. If the server is busy, an arriving (primary or repeated) customer involves into collision with customer under service and they both moves into the orbit. The retrial time of requests are exponentially distributed with rate σ/N. We assume that the server is unreliable, that is its lifetime is supposed to be exponentially distributed with failure rate γ_0 if the server is idle and with rate γ_1 if it is busy. When the server breaks down, it is immediately sent for repair and the recovery time is assumed to be exponentially distributed with rate γ_2. We deal with the case when the server is down all sources continue generation of customers and send it to the orbit, similarly customers may retry from the orbit to the server but all arriving customers immediately go into the orbit. Furthermore, in this unreliable model we suppose the interrupted request goes to the orbit immediately and its next service is independent of the interrupted one. All random variables involved in the model construction are assumed to be independent of each other.

Let $i(t)$ be the number of customers in the system at time t, that is, the total number of customers in orbit and in service. Similarly, let $k(t)$ be the server state at time t, that is

$$k(t) = \begin{cases} 0, & \text{if the server is idle,} \\ 1, & \text{if the server is busy,} \\ 2, & \text{if the server is down (under repair).} \end{cases}$$

Thus, we will investigate the process $\{k(t), i(t)\}$, which is not a Markov-process. To be a Markov one we will use method of supplementary variable, namely, we will consider two variants: the residual service time method and the elapsed service time method, and then we will compare them.

3 Method of Residual Service Time

Let us denote by $z(t)$ the random process, equal to the residual service time, that is time interval from the moment t until the end of successful service of the customer.

Thus, we will investigate the Markov process $\{k(t), i(t), z(t)\}$, which has a variable number of components, depending on the server state, since the component $z(t)$ is determined only in those moments when $k(t) = 1$.

Let us define the stationary probabilities as follows:

$$P_0(i) = P\{k(t) = 0, i(t) = i\},$$
$$P_1(i, z) = P\{k(t) = 1, i(t) = i, z(t) < z\},$$
$$P_2(i) = P\{k(t) = 2, i(t) = i\}.$$

To get $P_0(i)$, $P_1(i, z)$ and $P_2(i)$ the following system of Kolmogorov equations can be derived

$$
\begin{aligned}
&\frac{\partial P_1(1,0)}{\partial z} - [\lambda + \gamma_0] P_0(0) + \gamma_2 P_2(0) = 0, \\
&\frac{\partial P_1(1,z)}{\partial z} - \frac{\partial P_1(1,0)}{\partial z} - \left[\lambda \frac{N-1}{N} + \gamma_1\right] P_1(1,z) \\
&\qquad\qquad + \lambda B(z) P_0(0) + \frac{\sigma}{N} B(z) P_0(1) = 0, \\
&-[\lambda + \gamma_2] P_2(0) + \gamma_0 P_0(0) = 0, \\
&\frac{\partial P_1(i+1,0)}{\partial z} - \left[\lambda \frac{N-i}{N} + \gamma_0 + \frac{i}{N}\sigma\right] P_0(i) + \gamma_2 P_2(i) \\
&\qquad\qquad + \lambda \frac{N-i+1}{N} P_1(i-1) + \frac{i-1}{N}\sigma P_1(i) = 0, \\
&\frac{\partial P_1(i,z)}{\partial z} - \frac{\partial P_1(i,0)}{\partial z} - \left[\lambda \frac{N-i}{N} + \gamma_1 + \frac{i-1}{N}\sigma\right] P_1(i,z) \\
&\qquad\qquad + \lambda \frac{N-i+1}{N} P_0(i-1) B(z) + \frac{i}{N}\sigma P_0(i) B(z) = 0, \\
&-\left[\lambda \frac{N-i}{N} + \gamma_2\right] P_2(i) + \gamma_0 P_0(i) + \gamma_1 P_1(i) \\
&\qquad\qquad + \lambda \frac{N-i+1}{N} P_2(i-1) = 0.
\end{aligned} \tag{1}
$$

Let us introduce the partial characteristic functions

$$
H_k(u) = \sum_{i=0}^{N} e^{jui} P_k(i),\ k = 0, 2 \qquad H_1(u,z) = \sum_{i=1}^{N} e^{jui} P_1(i,z),
$$

where $j = \sqrt{-1}$ is imaginary unit, then system (1) can be rewritten as

$$
\begin{aligned}
&e^{-ju} \frac{\partial H_1(u,0)}{\partial z} + j \frac{(\sigma - \lambda)}{N} \frac{dH_0(u)}{du} + j \frac{(\lambda e^{ju} - \sigma)}{N} \frac{dH_1(u)}{du} \\
&\qquad\qquad - [\lambda + \gamma_0] H_0(u) + \left[\lambda e^{ju} - \frac{\sigma}{N}\right] H_1(u) + \gamma_2 H_2(u) = 0, \\
&\frac{\partial H_1(u,z)}{\partial z} - \frac{\partial H_1(u,0)}{\partial z} + j \frac{(\lambda e^{ju} - \sigma)}{N} B(z) \frac{dH_0(u)}{du} \\
&\quad + j \frac{(\sigma - \lambda)}{N} \frac{\partial H_1(u,z)}{\partial u} + \lambda e^{ju} B(z) H_0(u) - \left[\lambda + \gamma_1 - \frac{\sigma}{N}\right] H_1(u,z) = 0, \\
&j \frac{\lambda(e^{ju} - 1)}{N} \frac{dH_2(u)}{du} + \gamma_0 H_0(u) + \gamma_1 H_1(u) + \left[\lambda(e^{ju} - 1) - \gamma_2\right] H_2(u) = 0.
\end{aligned} \tag{2}
$$

Summarizing the equations of the system (2) and executing limiting transition under condition $z \to \infty$ we obtain equation in the form

$$-e^{-ju}\frac{\partial H_1(u,0)}{\partial z} + j\frac{\lambda}{N}\left[H_0^{'}(u) + H_1^{'}(u) + H_2^{'}(u)\right] \\ + \lambda\left[H_0(u) + H_1(u) + H_2(u)\right] = 0\ . \quad (3)$$

The solution of systems (2) and (3) for finite values N causes certain difficulties therefore we will find solution under condition of unlimited growing number of sources, that is $N \to \infty$.

3.1 Asymptotic Analysis

Theorem 1. *Let $i(t)$ be number of customers in a closed retrial queuing system M/GI/1//N with the collisions of customers and unreliable server, then*

$$\lim_{N\to\infty} \mathsf{E}\exp\left\{jw\frac{i(t)}{N}\right\} = \exp\left\{jw\kappa\right\}, \quad (4)$$

where value of parameter κ is the positive solution of the equation

$$(1-\kappa)\lambda - \delta(\kappa)\left[R_0(\kappa) - R_1(\kappa)\right] + \gamma_1 R_1(\kappa) = 0, \quad (5)$$

here $\delta(\kappa)$ is

$$\delta(\kappa) = (1-\kappa)\lambda + \sigma\kappa, \quad (6)$$

and the stationary distributions of probabilities $R_k(\kappa)$ of the service state k are determined as follows

$$R_0(\kappa) = \left\{\frac{\gamma_0+\gamma_2}{\gamma_2} + \frac{\gamma_1+\gamma_2}{\gamma_2}\cdot\frac{\delta(\kappa)}{\delta(\kappa)+\gamma_1}\left[1 - B^*(\delta(\kappa)+\gamma_1)\right]\right\}^{-1},$$

$$R_1(\kappa) = R_0(\kappa)\frac{\delta(\kappa)}{\delta(\kappa)+\gamma_1}\cdot\left[1 - B^*(\delta(\kappa)+\gamma_1)\right], \quad (7)$$

$$R_2(\kappa) = \frac{1}{\gamma_2}\left[\gamma_0 R_0(\kappa) + \gamma_1 R_1(\kappa)\right].$$

Proof. Denoting $\frac{1}{N} = \varepsilon$ and executing the following replacements in system (2)

$$u = \varepsilon w, \qquad H_k(u) = F_k(w,\varepsilon)\,, k = 0, 2; \qquad H_1(u,z) = F_1(w,z,\varepsilon),$$

we can write systems (2) and (3) in the form:

$$e^{-j\varepsilon w}\frac{\partial F_1(w,0,\varepsilon)}{\partial z}+j\left(\sigma-\lambda\right)\frac{\partial F_0(w,\varepsilon)}{\partial w}+j\left(\lambda e^{j\varepsilon w}-\sigma\right)\frac{\partial F_1(w,\varepsilon)}{\partial w}$$
$$-\left(\lambda+\gamma_0\right)F_0(w,\varepsilon)+\left[\lambda e^{j\varepsilon w}-\varepsilon\sigma\right]F_1(w,\varepsilon)+\gamma_2F_2(w,\varepsilon)=0,$$
$$\frac{\partial F_1(w,z,\varepsilon)}{\partial z}-\frac{\partial F_1(w,0,\varepsilon)}{\partial z}+j\left(\lambda e^{j\varepsilon w}-\sigma\right)B(z)\frac{\partial F_0(w,\varepsilon)}{\partial w}$$
$$+j\left(\sigma-\lambda\right)\frac{\partial F_1(w,z,\varepsilon)}{\partial w}+\lambda e^{j\varepsilon w}B(z)F_0(w,\varepsilon)$$
$$-\left[\lambda+\gamma_1-\varepsilon\sigma\right]F_1(w,z,\varepsilon)=0, \tag{8}$$
$$j\lambda\left(e^{j\varepsilon w}-1\right)\frac{\partial F_2(w,\varepsilon)}{\partial w}+\gamma_0F_0(w,\varepsilon)+\gamma_1F_1(w,\varepsilon)$$
$$+\left[\lambda\left(e^{j\varepsilon w}-1\right)-\gamma_2\right]F_2(w,\varepsilon)=0,$$
$$-e^{-j\varepsilon w}\frac{\partial F_1(w,0,\varepsilon)}{\partial z}+j\lambda\frac{\partial}{\partial w}\left[F_0(w,\varepsilon)+F_1(w,\varepsilon)+F_2(w,\varepsilon)\right]$$
$$+\lambda\left[F_0(w,\varepsilon)+F_1(w,\varepsilon)+F_2(w,\varepsilon)\right]=0\ .$$

Carrying out limiting transition under conditions $\varepsilon\to 0$, denoting $\lim\limits_{\varepsilon\to 0}F_k(w,\varepsilon)=F_k(w)$, $k=0,2$; $\lim\limits_{\varepsilon\to 0}F_1(w,z,\varepsilon)=F_1(w,z)$ system (8) can be rewritten as

$$\frac{\partial F_1(w,0)}{\partial z}+j\left(\sigma-\lambda\right)\frac{dF_0(w)}{dw}+j\left(\lambda-\sigma\right)\frac{dF_1(w)}{dw}-\left(\lambda+\gamma_0\right)F_0(w)$$
$$+\lambda F_1(w)+\gamma_2F_2(w)=0,$$
$$\frac{\partial F_1(w,z)}{\partial z}-\frac{\partial F_1(w,0)}{\partial z}+j\left(\lambda-\sigma\right)B(z)\frac{dF_0(w)}{dw}+j\left(\sigma-\lambda\right)\frac{\partial F_1(w,z)}{\partial w}$$
$$+\lambda B(z)F_0(w)-\left[\lambda+\gamma_1\right]F_1(w,z)=0, \tag{9}$$
$$\gamma_0F_0(w)+\gamma_1F_1(w)-\gamma_2F_2(w)=0,$$
$$-\frac{\partial F_1(w,0)}{\partial z}+j\lambda\frac{d}{dw}\left[F_0(w)+F_1(w)+F_2(w)\right]$$
$$+\lambda\left[F_0(w)+F_1(w)+F_2(w)\right]=0\ .$$

Let us write the solution of system (9) in product-form

$$F_k(w)=R_k\Phi(w),\qquad k=0,2;\qquad F_1(w,z)=R_1(z)\Phi(w), \tag{10}$$

where R_0, $R_1(z)$, R_2 are the limiting probability distributions of the server state k under conditions $N \to \infty$ and $\Phi(w)$ is limiting characteristic function of the stationary distribution of random process $\frac{i(t)}{N}$. Substituting this solution into (9), we obtain

$$R_1^{'}(0) + j\left(\sigma - \lambda\right)\left[R_0 - R_1\right]\frac{\partial\Phi(w)/\partial w}{\Phi(w)} - \left(\lambda + \gamma_0\right)R_0 + \lambda R_1 + \gamma_2 R_2 = 0,$$

$$R_1^{'}(z) - R_1^{'}(0) + j\left(\sigma - \lambda\right)\left[R_1(z) - R_0 B(z)\right]\frac{\partial\Phi(w)/\partial w}{\Phi(w)} + \lambda B(z) R_0 - \left[\lambda + \gamma_1\right]R_1(z) = 0, \tag{11}$$

$$\gamma_0 R_0 + \gamma_1 R_1 - \gamma_2 R_2 = 0,$$

$$j\lambda\frac{\partial\Phi(w)/\partial w}{\Phi(w)} + \lambda - R_1^{'}(0) = 0.$$

The above relations allows to write down this function in the following form

$$\Phi(w) = \exp\left(jw\kappa\right),$$

which coincides with equality (4). Using notation (6) and taking into account that $j\dfrac{\partial\Phi(w)/\partial w}{\Phi(w)} = -\kappa$, system (11) can be rewritten as

$$\begin{aligned}
&R_1^{'}(0) - \delta(\kappa)\left[R_0 - R_1\right] - \gamma_0 R_0 + \gamma_2 R_2 = 0,\\
&R_1^{'}(z) = R_1^{'}(0) + \left[\delta(\kappa) + \gamma_1\right]R_1(z) - \delta(\kappa) R_0 B(z) 0,\\
&\gamma_0 R_0 + \gamma_1 R_1 - \gamma_2 R_2 = 0,\\
&R_1^{'}(0) - \lambda(1 - \kappa) = 0.
\end{aligned} \tag{12}$$

Let us consider the second equation of the system (12) in more details. It can be proved that the solution of this equation has the form

$$R_1(z) = e^{[\delta(\kappa)+\gamma_1]z}\int\limits_0^z e^{-[\delta(\kappa)+\gamma_1]x}\left\{R_1^{'}(0) - \delta(\kappa)R_0 B(x)\right\}dx. \tag{13}$$

Executing the limiting transition at $z \to \infty$ and taking into account that the first factor of the right hand side of (13) in a limiting condition tends to infinity, we can conclude that the second factor will be equal to zero, that is

$$\int\limits_0^\infty e^{-[\delta(\kappa)+\gamma_1]x}\left\{R_1^{'}(0) - \delta(\kappa)R_0 B(x)\right\}dx = 0.$$

Performing simple transformations, we will obtain

$$R_1'(0) = \delta(\kappa) R_0 B^*(\delta(\kappa) + \gamma_1). \tag{14}$$

Now, let us add the first and third equations of system (12) and, taking into account the received equality (14), the system (12) can be rewritten in the form

$$\begin{aligned}
&R_1'(0) - \delta(\kappa) R_0 + [\delta(\kappa) + \gamma_1] R_1 = 0, \\
&R_1'(0) = \delta(\kappa) R_0 B^*(\delta(\kappa) + \gamma_1), \\
&\gamma_0 R_0 + \gamma_1 R_1 - \gamma_2 R_2 = 0, \\
&R_1'(0) - \lambda(1 - \kappa) = 0.
\end{aligned} \tag{15}$$

From the first three equations of system (15) and the normalization condition it is not difficult to obtain expressions for R_k, which coincides with (7) and, finally, equality (5) obviously follows from the first and fourth equations of system (15).

Theorem is proved. □

4 Method of Elapsed Service Time

Let us denote by $y(t)$ the supplementary random process, equal to the elapsed service time of the customer till the moment t.

It is obvious that $\{k(t), i(t), y(t)\}$ is Markov process. Let us note, $y(t)$ is defined only in those moments when the server is busy, that is, when $k(t) = 1$.

Define the stationary probabilities as

$$p_0(i) = P\{k(t) = 0, i(t) = i\},$$

$$p_1(i, y) = \frac{\partial P\{k(t) = 1, i(t) = i, y(t) < y\}}{\partial y},$$

$$p_2(i) = P\{k(t) = 2, i(t) = i\}.$$

To determine $p_0(i)$, $p_1(i,y)$ and $p_2(i)$ the following system of Kolmogorov equations can be written

$$
\begin{aligned}
&-\left[\lambda\frac{N-i}{N}+\frac{i}{N}\sigma+\gamma_0\right]p_0(i)+\int\limits_0^\infty p_1(i+1,y)\mu(y)dy\\
&\qquad\qquad+\lambda\frac{N-i+1}{N}p_1(i-1)+\frac{i-1}{N}\sigma p_1(i)+\gamma_2 p_2(i)=0,\\
&\frac{\partial p_1(i,y)}{\partial y}=-\left[\lambda\frac{N-i}{N}+\frac{i-1}{N}\sigma+\mu(y)+\gamma_1\right]p_1(i,y),\\
&-\left[\lambda\frac{N-i}{N}+\gamma_2\right]p_2(i)+\lambda\frac{N-i+1}{N}p_2(i-1)+\gamma_0 p_0(i)+\gamma_1 p_1(i)=0,
\end{aligned}
\tag{16}
$$

with boundary condition

$$
p_1(i,0)=\lambda\frac{N-i+1}{N}p_0(i-1)+\frac{i}{N}\sigma p_0(i). \tag{17}
$$

Introducing the partial characteristic functions

$$
H_k(u)=\sum_{i=0}^{N}e^{jui}p_k(i),\;\; k=0,2; \qquad H_1(u,y)=\sum_{i=1}^{N}e^{jui}p_1(i,y),
$$

system (16) and Eq. (17) we will rewrite in the form

$$
\begin{aligned}
&-(\lambda+\gamma_0)H_0(u)+\left[\lambda e^{ju}-\frac{\sigma}{N}\right]H_1(u)+e^{-ju}\int\limits_0^\infty H_1(u,y)\mu(y)dy\\
&\qquad\qquad+\gamma_2 H_2(u)+j\frac{(\sigma-\lambda)}{N}\frac{dH_0(u)}{du}+j\frac{(\lambda e^{ju}-\sigma)}{N}\frac{dH_1(u)}{du}=0,\\
&\frac{\partial H_1(u,y)}{\partial y}=\left[\frac{\sigma}{N}-\lambda-\mu(y)-\gamma_1\right]H_1(u,y)-j\frac{(\lambda-\sigma)}{N}\frac{\partial H_1(u,y)}{\partial u},\\
&\gamma_0 H_0(u)+\gamma_1 H_1(u)+\left[\lambda(e^{ju}-1)-\gamma_2\right]H_2(u)\\
&\qquad\qquad\qquad\qquad+j\frac{\lambda(e^{ju}-1)}{N}\frac{dH_2(u)}{du}=0,\\
&H_1(u,0)=\lambda e^{ju}H_0(u)+j\frac{(\lambda e^{ju}-\sigma)}{N}\frac{dH_0(u)}{du}.
\end{aligned}
\tag{18}
$$

4.1 Asymptotic Analysis

By using asymptotic methods for the first order solution to (18) we obtain

Theorem 2. *Let $i(t)$ be number of customers in a closed retrial queuing system M/GI/1//N with the collisions of customers and unreliable server, then*

$$\lim_{N\to\infty} \mathsf{E}\exp\left\{jw\frac{i(t)}{N}\right\} = \exp\left\{jw\kappa\right\}, \tag{19}$$

where value of parameter κ is the positive solution of the equation

$$(1-\kappa)\,\lambda - \delta(\kappa)\left[R_0(\kappa) - R_1(\kappa)\right] + \gamma_1 R_1(\kappa) = 0, \tag{20}$$

here $\delta(\kappa)$ is

$$\delta(\kappa) = (1-\kappa)\,\lambda + \sigma\kappa, \tag{21}$$

and the stationary distributions of probabilities $R_k(\kappa)$ of the service state k are defined as follows

$$\begin{aligned} R_0(\kappa) &= \left\{\frac{\gamma_0+\gamma_2}{\gamma_2} + \frac{\gamma_1+\gamma_2}{\gamma_2}\cdot\frac{\delta(\kappa)}{\delta(\kappa)+\gamma_1}\left[1 - B^*(\delta(\kappa)+\gamma_1)\right]\right\}^{-1}, \\ R_1(\kappa) &= R_0(\kappa)\frac{\delta(\kappa)}{\delta(\kappa)+\gamma_1}\cdot\left[1 - B^*(\delta(\kappa)+\gamma_1)\right], \\ R_2(\kappa) &= \frac{1}{\gamma_2}\left[\gamma_0 R_0(\kappa) + \gamma_1 R_1(\kappa)\right]. \end{aligned} \tag{22}$$

Proof. Denoting $\frac{1}{N} = \varepsilon$, in system (18) let us introduce the following substitutions

$$u = \varepsilon w, \qquad H_k(u) = F_k(w,\varepsilon)\,, k = 0,2; \qquad H_1(u,y) = F_1(w,y,\varepsilon),$$

then we will receive system of the equations

$$\begin{aligned} &-(\lambda+\gamma_0)F_0(w,\varepsilon) + \left[\lambda e^{j\varepsilon w} - \varepsilon\sigma\right]F_1(w,\varepsilon) + e^{-j\varepsilon w}\int_0^\infty F_1(w,y,\varepsilon)\mu(y)dy \\ &\qquad + \gamma_2 F_2(w,\varepsilon) + j(\sigma-\lambda)\frac{\partial F_0(w,\varepsilon)}{\partial w} + j(\lambda e^{j\varepsilon w} - \sigma)\frac{\partial F_1(w,\varepsilon)}{\partial w} = 0, \\ &\frac{\partial F_1(w,y,\varepsilon)}{\partial y} = \left[\varepsilon\sigma - \lambda - \mu(y) - \gamma_1\right]F_1(w,y,\varepsilon) - j(\lambda-\sigma)\frac{\partial F_1(w,y,\varepsilon)}{\partial w}, \\ &\gamma_0 F_0(w,\varepsilon) + \gamma_1 F_1(w,\varepsilon) + \left[\lambda(e^{j\varepsilon w}-1) - \gamma_2\right]F_2(w,\varepsilon) \\ &\qquad + j\lambda(e^{j\varepsilon w}-1)\frac{\partial F_2(w,\varepsilon)}{\partial w} = 0, \\ &F_1(w,0,\varepsilon) = \lambda e^{j\varepsilon w}F_0(w,\varepsilon) + j(\lambda e^{j\varepsilon w} - \sigma)\frac{\partial F_0(w,\varepsilon)}{\partial w}. \end{aligned} \tag{23}$$

Taking the limiting transition under conditions $\varepsilon \to 0$ let us denote $\lim\limits_{\varepsilon\to 0} F_k(w,\varepsilon) = F_k(w),\ k = 0,2;\ \lim\limits_{\varepsilon\to 0} F_1(w,y,\varepsilon) = F_1(w,y)$. Then system (23) can be rewritten as

$$-(\lambda+\gamma_0)F_0(w) + \lambda F_1(w) + \gamma_2 F_2(w) + \int\limits_0^\infty F_1(w,y)\mu(y)dy$$
$$+ j(\lambda-\sigma)\left[\frac{dF_1(w)}{dw} - \frac{dF_0(w)}{dw}\right] = 0\ ,$$
$$\frac{\partial F_1(w,y)}{\partial y} = -\left[\lambda+\mu(y)+\gamma_1\right]F_1(w,y) - j(\lambda-\sigma)\frac{\partial F_1(w,y)}{\partial w}, \qquad (24)$$
$$\gamma_0 F_0(w) + \gamma_1 F_1(w) - \gamma_2 F_2(w) = 0,$$
$$F_1(w,0) = \lambda F_0(w) + j(\lambda-\sigma)\frac{dF_0(w)}{dw}.$$

The solution of the system (24) can be written in product-form

$$F_k(w) = R_k\varPsi(w),\ k = 0,2; \qquad F_1(w,y) = R_1(y)\varPsi(w). \qquad (25)$$

Substituting this solution into (24) we will receive

$$\int\limits_0^\infty R_1(y)\mu(y)dy - \lambda\left(R_0 - R_1\right) - \gamma_0 R_0 + \gamma_2 R_2$$
$$+ j(\lambda-\sigma)(R_1 - R_0)\frac{\partial\varPsi(w)/\partial w}{\varPsi(w)} = 0,$$
$$R_1^{'}(y) = -\left[\lambda+\mu(y)+\gamma_1\right]R_1(y) - j(\lambda-\sigma)R_1(y)\frac{\partial\varPsi(w)/\partial w}{\varPsi(w)}, \qquad (26)$$
$$\gamma_0 R_0 + \gamma_1 R_1 - \gamma_2 R_2 = 0,$$
$$R_1(0) = \lambda R_0 + j(\lambda-\sigma)R_0\frac{\partial\varPsi(w)/\partial w}{\varPsi(w)}.$$

from which it follows, that function $\varPsi(w)$ has the form

$$\varPsi(w) = \exp\left(jw\kappa\right), \qquad (27)$$

coinciding with equality (19). Using the notation (21) the system (26) can be rewritten as

$$
\begin{aligned}
&\int_0^\infty R_1(y)\mu(y)dy = \delta(\kappa)\left(R_0 - R_1\right) + \gamma_0 R_0 - \gamma_2 R_2,\\
&R_1^{'}(y) = -\left[\delta(\kappa) + \mu(y) + \gamma_1\right] R_1(y),\\
&\gamma_0 R_0 + \gamma_1 R_1 - \gamma_2 R_2 = 0,\\
&R_1(0) = \delta(\kappa) R_0.
\end{aligned}
\tag{28}
$$

Let us consider the second equation of system (28) in more details. It is not difficult to obtain a solution of this equation, taking the fourth equality of system (28) as the initial condition, and as a result we get

$$R_1(y) = \delta(\kappa) R_0 \left[1 - B(y)\right] e^{-[\delta(\kappa)+\gamma_1]y}. \tag{29}$$

To find R_1 integrate equality (29) with respect to y from 0 to ∞ and receive an expression in the form

$$R_1 = R_0 \frac{\delta(\kappa)}{\delta(\kappa) + \gamma_1} \cdot \left[1 - B^*(\delta(\kappa) + \gamma_1)\right]. \tag{30}$$

Expression for R_2 obviously follows from the third equation of system (28)

$$R_2 = \frac{1}{\gamma_2}\left[\gamma_0 R_0 + \gamma_1 R_1\right], \tag{31}$$

and, finally, from equalities (30) and (31), keeping in mind the normalization condition for R_0 we have

$$R_0 = \left\{\frac{\gamma_0 + \gamma_2}{\gamma_2} + \frac{\gamma_1 + \gamma_2}{\gamma_2} \cdot \frac{\delta(\kappa)}{\delta(\kappa) + \gamma_1}\left[1 - B^*(\delta(\kappa) + \gamma_1)\right]\right\}^{-1}.$$

Thus, we have determined R_0, R_1 and R_2 that coincides with equalities (22).

Let us return to system (24). Integrating the second equation of the system with respect to y from 0 to ∞, adding it with other equations of system (24), substituting the decomposition (25) and taking into account the explicit form (27) of the function $\Psi(w)$, we obtain an equation in the form

$$\int_0^\infty R_1(y)\mu(y)dy = \lambda(1 - \kappa). \tag{32}$$

From (32) and the first and third equations of the system (28) it is obviously follows Eq. (20) for κ.

Theorem is proved. □

5 Comparison of the Methods of Residual and Elapsed Time

At a research of the closed retrial queuing system M/GI/1//N with collision of customers and unreliable server by asymptotic analysis for a Markovization of process $\{k(t), i(t)\}$ two methods were considered: the method of elapsed service time and the method of residual service time. From the Theorems 1 and 2 it follows, as it was expected, that the basic characteristics of the system, such as the stationary probability distribution R_k of the server states k and the asymptotic average κ of the normalized number of customers in the system are the same and do not depend on the method of investigation. Of course, it should be so, since only the proofs are different.

Let us note that the use of the elapsed service time method is necessary for a further research of number of transitions of a customer into the orbit, and also for a further research of the sojourn time of a customer in the orbit.

The residual service time method is used for finding the probability distribution of the number of customers in the system and also it is necessary at a further research of the mean sojourn time of a customer under service.

6 Conclusion

In this paper, a finite-source retrial queuing system M/GI/1 with collisions of customers and unreliable server was considered. Two methods of an supplementary variable was presented: method of elapsed service time and method of residual service time. The research of system has been conducted by an asymptotic analysis under condition of unlimited growing number of sources. As a result of the investigation the first order approximations of the basic characteristics of the system, such as a stationary probability distribution of the server states and the asymptotic average of the normalized number of customers in the system was obtained. It was shown, as it was expected, that specified characteristics are the same and do not depend on a type of applied method of a supplementary variable. In addition, advantages for using each of the considered methods were given and the necessity of their application for further researches of the system was indicated.

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