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An algorithmic approach to analysing the reliability of a controllable unreliable queue with two heterogeneous servers[☆]

Dmitry Efrosinin[a,b,*], Janos Sztrik[c]

[a] Institute for Stochastics, Johannes Kepler University, Altenbergerstrasse 69, Linz 4040, Austria
[b] Peoples' Friendship University of Russia (RUDN University), Miklukho-Maklaya street 6, Moscow 117198, Russia
[c] University of Debrecen, Egyetem ter 1, Debrecen 4032, Hungary

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ABSTRACT

We consider a Markovian queueing system with two unreliable heterogeneous servers and one common queue. The servers serve customers without preemption and fail only if they are busy. Customers are allocated to one or the other server via a threshold control policy which prescribes using the faster server whenever it is free and the slower server only when the number of waiting customers exceeds a specified threshold level that depends on the state of the faster server. This paper focuses on the reliability analysis of a system with unreliable heterogeneous servers. First, we obtain the stationary state distribution using a matrix-geometric solution method. Second, we analyse the lifetimes of the servers and of the system. We provide algorithms for calculating the stationary reliability characteristics, reliability functions in terms of the Laplace transform and the mean times to the first failure. A new reliability measure is introduced in the form of the discrete distribution function of the number of failures during a specified life time that is derived from a probability generating function. The effects of various parameters on these reliability characteristics are analysed numerically.

1. Introduction

To improve modern communication systems in terms of performance and reliability, they can be supplied with controllable heterogeneous environment. The heterogeneity in such systems may be easily explained by virtue of the following examples. The data centers with a cloud computing paradigm containing the execution servers of many generations as a consequence of continuous system updates (Bai, Xi, Zhu, & Huang, 2015). Obviously in this system the servers can differ in terms of speed, capacity, availability, power consumption an so on. Another example is a hybrid wireless channel working on the basis of Radio Frequency/Free Space Optic (RF/FSO) technology (Vishnevskii, Semenova, & Sharov, 2013). The links of this channel have unequal data throughput, availability and reliability characteristics. The capacity of RF link is constrained by limits to link throughputs on the order of 10 s of Mbps. On the contrary, the commercial FSO currently provide throughputs of several Gbts but the link availability is limited by adverse weather conditions like fogs and heavy snowfalls. Therefore, the hybrid channel combines advantages of both types of links. One more example is a single cell of a cellular (3GPP LTE) network with a Licence Shared Access (LSA) technology, for details see Gudkova et al. (2015), which assumes that the band can be used when the owner does not need it. In this case heterogeneous environment consists of the reliable main and unreliable reserve pool of servers which is used according to a specified hysteretic control policy. The proposed examples have motivated us to apply the queueing system with unreliable heterogeneous servers for modelling the dynamic behaviour and analysis the relationships between different factors influencing on reliability of communication systems with heterogeneous unreliable environment.

Analyses of multi-server queueing systems generally assume that the servers are homogeneous. Mitrany and Avi-Itzhak (1967) and Neuts and Lucantoni (1979) studied the $M/M/s$ queueing system with server breakdowns and repairs. Levy and Yechiali (1976) analysed the $M/M/s$ queue with server vacation. A recent paper by Efrosinin, Samouylov, and Gudkova (2016) reported on stationary analysis of the busy period for a multi-server Markovian queueing system with simultaneous failures of servers. Queues with heterogeneous unreliable servers have rarely been addressed by research. A queueing system with two heterogeneous servers

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* Corresponding author at: Institute for Stochastics, Johannes Kepler University, Altenbergerstrasse 69, 4040 Linz, Austria.

E-mail addresses: dmitry.efrosinin@jku.at, dmitriy_e@mail.ru (D. Efrosinin), sztrik.janos@inf.unideb.hu (J. Sztrik).

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and multiple vacations was studied by Kumar and Madheswari (2005), who obtained the stationary queue length distribution by using a matrix geometric method and provided an analysis of busy period and waiting time. In Kumar, Madheswari, and Venkatakrishnan (2007), the same authors introduced the $M/M/2$ queueing system with heterogeneous servers subject to catastrophes, and provided a transient solution for the system under study. A heterogeneous two-server queueing system with balking and server breakdowns was studied by Yue, Yue, Yu, and Tian (2009). They used a matrix-geometric solution method to obtain some mean performance measures.

In a heterogeneous queueing system with one common queue, particularly in the case of service without preemption (a customer can not change the server during a service time) a mechanism that allocates customers to the servers must be specified. The majority of heterogeneous systems investigated use heuristic service policies (e.g. the Fastest Free Server (FFS) or Random Service Selection (RSS) policies). In fact, these policies are not optimal, if, for instance, the mean response time is to be minimized. As previously shown (see, e.g. the results of B & Jouini, 2016; Efrosinin, 2008; Koole, 1995; Lin & Kumar, 1984; Rykov & Efrosinin, 2009), the optimal allocation policy for heterogeneous queueing systems is one of a class of threshold policies where the less effective server is to be used only if the number of customers in the queue has reached some pre-specified threshold level. This result was confirmed for a queueing system with faster unreliable server and absolutely reliable slower server in Efrosinin (2013), Ozkan and Kharoufeh (2014) and for two unreliable heterogeneous servers in a system with constant retrial discipline in Efrosinin and Sztrik (2016). In the last paper mentioned, it was shown that for a fixed threshold policy the corresponding Markov process is of the QBD (quasi-birth-and-death) type with a tri-diagonal block infinitesimal matrix with a large number of bounding states.

While first steps in performance analyses of controllable heterogeneous queueing systems with completely reliable servers have already been published, application to heterogeneous models also requires a reliability analysis of such queues when servers are subject to failure. Here we use a forward-elimination-backward-substitution method expressed in matrix form in terms of the Laplace–Stiltjes transforms (LST) combined with probability generating function (PGF) approach to evaluate reliability measures such as reliability function (i.e., the complementary cumulative distribution function of the lifetime) and mean time to first failure for each server separately and for the group of servers under the fixed threshold allocation control policy. The reliability functions are obtained in terms of the Laplace transform (LT), and a numerical inversion algorithm is used to obtain the time-dependent functions. Additionally, we introduce a new discrete reliability metric in the form of the distribution of the number of failures during a certain lifetime. We expect that our results can be generalized to the case of an arbitrary controllable unreliable queueing model with a QBD structure.

The remainder of paper is organized as follows: In Section 2, we describe the mathematical model and present the stationary state distribution using a matrix-geometric solution method. In Section 3, we develop a computational analysis of the stationary reliability characteristics, the reliability function and the mean time to first failure. The number of failures during a certain life time is investigated in Section 4. In Section 5, numerical examples are provided to highlight the effect of some parameters on the reliability characteristics.

Hereafter, the notations $\mathbf{e}(n)$, $\mathbf{e}_j(n)$, and I_n are used respectively for the column vector consisting of 1's, the column vector with 1 in the jth (beginning from 0th) position and 0 elsewhere, and an identity matrix of the dimension n. When there is no need to emphasize the dimensions of these vectors, the suffix is omitted and dimensionality is determined by the context. The expressions $diag(a_1, \dots, a_n)$, $diag^+(a_1, \dots, a_n)$, and $diag^-(a_1, \dots, a_n)$ denote respectively the diagonal matrix, the upper diagonal matrix, and the lower diagonal matrix with entries $a_1, \dots, a_n$ that can be scalars or matrices.

2. Mathematical model and stationary distribution

In this paper, we address a two-server heterogeneous unreliable queueing model of the $M/M/2$ type as illustrated in Fig. 1(a).

Customers arrive according to a Poisson process with arrival rate λ. The service times are exponentially distributed with rates μ_1 and μ_2, where $\mu_1 \geq \mu_2$. We assume that the servers fail respectively at exponential rates α_1 and α_2. A server can fail only if it is busy. A failed server is repaired immediately, and the time required to repair it is exponentially distributed respectively with rates β_1 and β_2. A customer being served at the moment of failure is left at this server during repair and can be served when the server becomes operational again. The mechanism of allocation to the two servers is based on a threshold policy: Depending on the state of the faster server, the slower is used whenever the number of customers in the queue exceeds a certain threshold level.

Let $Q(t)$ and $D(t) = \{D_1(t), D_2(t)\}$ denote, respectively, the number of customers in the queue and the vector state of servers at time t, where service process

$$D_j(t) = \begin{cases} 0, & \text{the server } j \text{ is idle,} \\ 1, & \text{the server } j \text{ is busy and operational,} \\ 2, & \text{the server } j \text{ has failed.} \end{cases}$$

with transitions as shown in Fig. 1(b). The threshold policy $f = (q_1, q_2)$ is defined by two threshold levels $1 \leq q_2 \leq q_1 < \infty$. According to this policy, server 1 must be used upon new arrival whenever it is free and there are customers in the queue, whereas idle server 2 is ready to serve the arriving customers only if server 1 is in state 1 or 2 and the number of customers in the queue has reached the corresponding threshold value q_1 or q_2. If server 1 is in state 1 or 2 upon service completion at server 2 and the number of customers in the queue is smaller than q_1 or q_2, then further allocation of customers to server 2 is not possible. For the fixed threshold policy f the process

$$\{X(t)\}_{t\geq 0} = \{Q(t), D(t)\}_{t\geq 0} \tag{1}$$

is a continuous-time Markov chain with a state space given by

$$E = \{x = (q, d_1, d_2);\, q \in \mathbb{N}_0,\, (d_1, d_2) \in E_D\}, \tag{2}$$

where E_D is a set of states of servers that is defined as

$$E_D = \left\{ (d_1, d_2); \begin{array}{c} d_j \in \{0,1,2\},\, j \in \{1,2\},\, q = 0, \\ d_1 \in \{1,2\},\, d_2 \in \{0,1,2\},\, 1 \leq q \leq q_2 - 1, \\ d_1 \in \{1,2\},\, d_2 \in \{0,1,2\},\, (d_1, d_2) \neq (2,0), \\ q_2 \leq q \leq q_1 - 1, \\ d_j \in \{1,2\},\, j \in \{1,2\},\, q \geq q_1, \end{array} \right\}.$$

Next we partition E into blocks as follows:

$$(\mathbf{0}, \mathbf{0}) = \{(0, 0, d_2);\, d_2 \in \{0,1,2\}\},$$

$$(\mathbf{q}, \mathbf{1}) = \begin{cases} \{(q,1,0), (q,2,0), (q,1,1), (q,2,1), (q,1,2), (q,2,2)\}, & 0 \leq q \leq q_2 - 1, \\ \{(q,1,0), (q,1,1), (q,2,1), (q,1,2), (q,2,2)\}, & q_2 \leq q \leq q_1 - 1, \\ \{(q,1,1), (q,2,1), (q,1,2), (q,2,2)\}, & q \geq q_1. \end{cases}$$

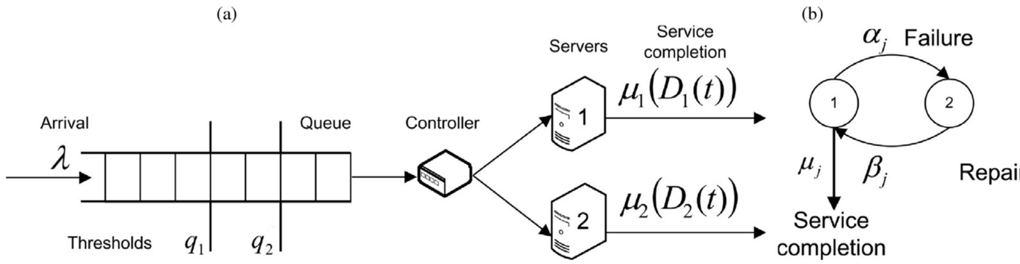

Fig. 1. Scheme of the queueing system $M/M/2$ (a) and transitions of the service process $D_j(t)$ (b).

Due to this notation, the infinitesimal generator of the Markov chain $\{X(t)\}_{t\geq 0}$ has the block-tridiagonal structure,

$$\Lambda = [\lambda_{xy}]_{x,y\in E} = diag(Q_{1,0}, \underbrace{Q_{1,1}, \dots, Q_{1,1}}_{q_2-1}, Q_{1,2}, \underbrace{Q_{1,3}, \dots, Q_{1,3}}_{q_1-q_2-1}, Q_{1,4}, Q_{1,5}, \dots)+$$

$$+\, diag^{+}(Q_{0,1}, \underbrace{Q_{0,2}, \dots, Q_{0,2}}_{q_2-1}, Q_{0,3}, \underbrace{Q_{0,4}, \dots, Q_{0,4}}_{q_1-q_2-1}, Q_{0,5}, Q_{0,6}, \dots)+$$

$$+\, diag^{-}(Q_{2,1}, \underbrace{Q_{2,2}, \dots, Q_{2,2}}_{q_2-1}, Q_{2,3}, \underbrace{Q_{2,4}, \dots, Q_{2,4}}_{q_1-q_2-1}, Q_{2,5}, Q_{2,6}, \dots).$$

The square matrices $Q_{1,\,n}$, $0\leq n\leq 5$, include the transition rates inside the current block of states for a certain queue length q,

$$Q_{1,0} = \begin{pmatrix} -\lambda & 0 & 0 \\ \mu_2 & -(\lambda+\alpha_2+\mu_2) & \alpha_2 \\ 0 & \beta_2 & -(\lambda+\beta_2) \end{pmatrix},$$

$$Q_{1,1} = \begin{pmatrix} -(\lambda+\mu_1+\alpha_1) & \alpha_1 & 0 & 0 & 0 & 0 \\ \beta_1 & -(\lambda+\beta_1) & 0 & 0 & 0 & 0 \\ \mu_2 & 0 & -(\lambda+\mu+\alpha) & \alpha_1 & \alpha_2 & 0 \\ 0 & \mu_2 & \beta_1 & -(\lambda+\alpha_2+\beta_1+\mu_2) & 0 & \alpha_2 \\ 0 & 0 & \beta_2 & 0 & -(\lambda+\alpha_1+\beta_2+\mu_1) & \alpha_1 \\ 0 & 0 & 0 & \beta_2 & \beta_1 & -(\lambda+\beta) \end{pmatrix},$$

$$Q_{1,2} = Q_{1,1} + \lambda \mathbf{e}_1(6)\otimes \mathbf{e}'_3(6),$$

$$Q_{1,3} = \begin{pmatrix} -(\lambda+\mu_1+\alpha_1) & 0 & 0 & 0 & 0 \\ \mu_2 & -(\lambda+\mu+\alpha) & \alpha_1 & \alpha_2 & 0 \\ 0 & \beta_1 & -(\lambda+\alpha_2+\beta_1+\mu_2) & 0 & \alpha_2 \\ 0 & \beta_2 & 0 & -(\lambda+\alpha_1+\beta_2+\mu_1) & \alpha_1 \\ 0 & 0 & \beta_2 & \beta_1 & -(\lambda+\beta) \end{pmatrix},$$

$$Q_{1,4} = Q_{1,3} + \lambda \mathbf{e}_0(5)\otimes \mathbf{e}'_1(5),$$

$$Q_{1,5} = \begin{pmatrix} -(\lambda+\mu+\alpha) & \alpha_1 & \alpha_2 & 0 \\ \beta_1 & -(\lambda+\alpha_2+\beta_1+\mu_2) & 0 & \alpha_2 \\ \beta_2 & 0 & -(\lambda+\alpha_1+\beta_2+\mu_1) & \alpha_1 \\ 0 & \beta_2 & \beta_1 & -(\lambda+\beta) \end{pmatrix}.$$

The rectangular matrices $Q_{0,\,n}$, $1\leq n\leq 6$, include the transition rates from the block of states with queue length q to the block with queue length $q+1$,

$$Q_{0,1} = \lambda\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},\quad Q_{0,3} = \lambda\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},\quad Q_{0,5} = \lambda\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$Q_{0,2} = \lambda I_6,\quad Q_{0,4} = \lambda I_5,\quad Q_{0,6} = \lambda I_4, \mu = \mu_1+\mu_2, \alpha = \alpha_1+\alpha_2, \beta = \beta_1+\beta_2.$$

The rectangular matrices $Q_{2,\,n}$, $1\leq n\leq 6$, include the transition rates from the block of states with queue length q to the block with queue length $q-1$,

$$Q_{2,1} = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mu_1 \\ 0 & 0 & 0 \end{pmatrix},\quad Q_{2,2} = \begin{pmatrix} \mu_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},\quad Q_{2,3} = \begin{pmatrix} \mu_1 & 0 & 0 & \alpha_1 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$Q_{2,4}=\begin{pmatrix}\mu_1&0&\alpha_1&0&0\\0&\mu_1&0&0&0\\0&0&\mu_2&0&0\\0&0&0&\mu_1&0\\0&0&0&0&0\end{pmatrix},\quad Q_{2,5}=\begin{pmatrix}0&\mu&0&0&0\\0&0&\mu_2&0&0\\0&0&0&\mu_1&0\\0&0&0&0&0\end{pmatrix},\quad Q_{2,6}=\begin{pmatrix}\mu&0&0&0\\0&\mu_2&0&0\\0&0&\mu_1&0\\0&0&0&0\end{pmatrix}.$$

Let us denote by $\boldsymbol{\pi}=(\boldsymbol{\pi}_{0,0},\boldsymbol{\pi}_{0,1},\boldsymbol{\pi}_{1,1},\boldsymbol{\pi}_{2,1},\dots)$ the stationary probability vector of Λ which satisfies

$$\boldsymbol{\pi}\Lambda=\mathbf{0},\ \boldsymbol{\pi}\mathbf{e}=1. \tag{3}$$

Computation of the stationary state distribution is reduced to solving a block-tridiagonal system. The process $\{X(t)\}_{t\geq 0}$ is in the format of a quasi-birth-and-death (QBD) process, which allows a matrix-analytic approach to be applied. Based on (Neuts, 1981, Theorem 3.1.1), the stationary probability vector $\boldsymbol{\pi}$ of the QBD process exists if and only if

$$\mathbf{p}Q_{0,6}\mathbf{e}(4)<\mathbf{p}Q_{2,6}\mathbf{e}(4),$$

where $\mathbf{p}=(p_1,p_2,p_3,p_4)$ is the invariant probability of the matrix $Q_{0,6}+Q_{1,5}+Q_{2,6}$. This vector can be obtained by solving the system $\mathbf{p}(Q_{0,6}+Q_{1,5}+Q_{2,6})=\mathbf{0}$ and $\mathbf{pe}(4)=1$. After some routine manipulation, we obtain the condition

$$\rho=\frac{\lambda}{\sum_{j=1}^{2}\frac{\beta_j\mu_j}{\alpha_j+\beta_j}}<1. \tag{4}$$

According to the last condition, to guarantee the existence of stationary regime the arrival rate must be less than the sum of efficient service rates. Here the value $\left(\frac{\beta_j\mu_j}{\alpha_j+\beta_j}\right)^{-1}$ represents the mean time a customer spend on server j before it leaves the system.

Theorem 1. *The vectors of stationary probabilities $\boldsymbol{\pi}_{q,\,i}$, $q\geq 0$, can be computed as follows,*

$$\boldsymbol{\pi}_{0,0}=\boldsymbol{\pi}_{q_1,1}\prod_{j=0}^{q_1}M_{q_1-j}, \tag{5}$$

$$\boldsymbol{\pi}_{q,1}=\boldsymbol{\pi}_{q_1,1}\prod_{j=0}^{q_1-q-1}M_{q_1-j},\ 0\leq q\leq q_1-1,$$

$$\boldsymbol{\pi}_{q,1}=\boldsymbol{\pi}_{q_1,1}R^{q-q_1},\ q\geq q_1,$$

where the matrices M_i, $0\leq i\leq q_1$, are recursively defined

$$M_0=-Q_{2,1}Q_{1,0}^{-1},\ M_1=-Q_{2,2}(M_0Q_{0,1}+Q_{1,1})^{-1}, \tag{6}$$

$$M_q=-Q_{2,2}(M_{q-1}Q_{0,2}+Q_{1,1})^{-1},\ 2\leq q\leq q_2-1,$$

$$M_{q_2}=-Q_{2,3}(M_{q_2-1}Q_{0,2}+Q_{1,2})^{-1},\ M_{q_2+1}=-Q_{2,4}(M_{q_2}Q_{0,3}+Q_{1,3})^{-1},$$

$$M_q=-Q_{2,4}(M_{q-1}Q_{0,4}+Q_{1,3})^{-1},\ q_2+2\leq q\leq q_1-1,$$

$$M_{q_1}=-Q_{2,5}(M_{q_1-1}Q_{0,4}+Q_{1,4})^{-1}.$$

The vector $\boldsymbol{\pi}_{q_1,1}$ is a unique solution of the system of equations

$$\boldsymbol{\pi}_{q_1,1}\Bigg[\sum_{q=-1}^{q_1-1}\prod_{j=0}^{q_1-q-1}M_{q_1-j}+(I-R)^{-1}\Bigg]\mathbf{e}(4)=1, \tag{7}$$

$$\boldsymbol{\pi}_{q_1,1}(M_{q_1}Q_{0,5}+Q_{1,5}+RQ_{2,6})=\mathbf{0}.$$

The matrix R is a minimal solution of the matrix quadratic equation,

$$R^2Q_{2,6}+RQ_{1,5}+Q_{0,6}=0. \tag{8}$$

Proof. The last row of (5) and equation $R^2Q_{2,6}+RQ_{1,5}+Q_{0,6}=0$ follow from the properties of the QBD process (Neuts, 1981). If the stability condition holds, then (3) yields the system,

$$\boldsymbol{\pi}_{0,0}Q_{1,0}+\boldsymbol{\pi}_{0,1}Q_{2,1}=\mathbf{0},$$

$$\boldsymbol{\pi}_{q-1,1}Q_{0,1}+\boldsymbol{\pi}_{q,1}Q_{1,1}+\boldsymbol{\pi}_{q+1,1}Q_{2,2}=\mathbf{0},\ 2\leq q\leq q_2-1,$$

$$\boldsymbol{\pi}_{q_2-1,1}Q_{0,2}+\boldsymbol{\pi}_{q_2,1}Q_{1,2}+\boldsymbol{\pi}_{q_2+1,1}Q_{2,3}=\mathbf{0},$$

$$\boldsymbol{\pi}_{q_2,1}Q_{0,3}+\boldsymbol{\pi}_{q_2+1,1}Q_{1,3}+\boldsymbol{\pi}_{q_2+2,1}Q_{2,4}=\mathbf{0},$$

$$\boldsymbol{\pi}_{q-1,1}Q_{0,4}+\boldsymbol{\pi}_{q,1}Q_{1,3}+\boldsymbol{\pi}_{q+1,1}Q_{2,4}=\mathbf{0},\ q_2+2\leq q\leq q_1-1,$$

$$\boldsymbol{\pi}_{q_1-1,1}Q_{0,4}+\boldsymbol{\pi}_{q_1}Q_{1,4}+\boldsymbol{\pi}_{q_1+1}Q_{2,5}=\mathbf{0},$$

$$\boldsymbol{\pi}_{q_1,1}R^{q-q_1-1}Q_{0,5}+\boldsymbol{\pi}_{q_1,1}R^{q-q_1}Q_{1,5}+\boldsymbol{\pi}_{q_1,1}R^{q-q_1+1}Q_{2,6}=\mathbf{0},\ q\geq q_1+1.$$

The routine of substitution applied to the previous system leads to recursive relations,

$$\boldsymbol{\pi}_{0,0}=\boldsymbol{\pi}_{0,1}M_0, \tag{9}$$

$$\boldsymbol{\pi}_{q,1}=\boldsymbol{\pi}_{q+1,1}M_{q+1},\ 1\leq q\leq q_1-1,$$

where M_q is defined by (6), which implies the first two rows of (5). Finally, the vector $\boldsymbol{\pi}_{q_1,1}$ is obviously a unique solution of the system of Eq. (7), which consists of the normalizing condition and the balance equation for the probability vector $\boldsymbol{\pi}_{q_1,1}$ of the boundary states. □

3. Reliability characteristics of the system and servers

In this section we consider some reliability quantities of the system and servers. Let us denote by

$$A_1(t) = \mathbb{P}[X(t) = (q, d_1, d_2); d_1 \neq 2 \vee d_2 \neq 2],$$
$$A_2(t) = \mathbb{P}[X(t) = (q, d_1, d_2); d_1 \neq 2 \wedge d_2 \neq 2],$$
$$A_3(t) = \mathbb{P}[X(t) = (q, d_1, d_2); d_1 \neq 2],$$
$$A_4(t) = \mathbb{P}[X(t) = (q, d_1, d_2); d_2 \neq 2],$$

the pointwise availabilities of the system and servers. The stationary availability in the case n, $1 \le n \le 4$, is defined as $A_n = \lim_{t\to\infty} A_n(t)$.

Corollary 1. *The stationary availability can be computed by*

$$A_n = \boldsymbol{\pi}_{0,0}\mathbf{x}_{n,1} + \sum_{q=0}^{q_2-1} \boldsymbol{\pi}_{q,1}\mathbf{x}_{n,2} + \sum_{q=q_2}^{q_1-1} \boldsymbol{\pi}_{q,1}\mathbf{x}_{n,3} + \boldsymbol{\pi}_{q_1,1}(I-R)^{-1}\mathbf{x}_{n,4},\ 1 \le n \le 4,$$

where $A_2 = A_3 + A_4 - A_1$ and

$$\mathbf{x}_{1,1} = \mathbf{e}(3), \mathbf{x}_{1,2} = \sum_{k=0}^{4} \mathbf{e}_k(6), \mathbf{x}_{1,3} = \sum_{k=0}^{3} \mathbf{e}_k(5), \mathbf{x}_{1,4} = \sum_{k=0}^{2} \mathbf{e}_k(4),$$

$$\mathbf{x}_{2,1} = \sum_{k=0}^{1} \mathbf{e}_k(3), \mathbf{x}_{2,2} = \sum_{k=0}^{1} \mathbf{e}_{2k}(6), \mathbf{x}_{2,3} = \sum_{k=0}^{1} \mathbf{e}_k(5), \mathbf{x}_{2,4} = \mathbf{e}_0(4),$$

$$\mathbf{x}_{3,1} = \mathbf{e}(3), \mathbf{x}_{3,2} = \sum_{k=0}^{2} \mathbf{e}_{2k}(6), \mathbf{x}_{3,3} = \mathbf{e}_0 + \sum_{k=0}^{1} \mathbf{e}_{2k+1}(5), \mathbf{x}_{3,4} = \sum_{k=0}^{1} \mathbf{e}_{2k}(4),$$

$$\mathbf{x}_{4,1} = \sum_{k=0}^{1} \mathbf{e}_k(3), \mathbf{x}_{4,2} = \sum_{k=0}^{3} \mathbf{e}_k(6), \mathbf{x}_{4,3} = \sum_{k=0}^{2} \mathbf{e}_k(5), \mathbf{x}_{4,4} = \sum_{k=0}^{1} \mathbf{e}_k(4).$$

Corollary 2. *The stationary failure frequency of the server $l \in \{1, 2\}$ can be computed by*

$$B_l = \alpha_l \boldsymbol{\pi}_{0,0}\mathbf{y}_{l,1} + \sum_{q=0}^{q_2-1} \boldsymbol{\pi}_{q,1}\mathbf{y}_{l,2} + \sum_{q=q_2}^{q_1-1} \boldsymbol{\pi}_{q,1}\mathbf{y}_{l,3} + \boldsymbol{\pi}_{q_1,1}(I-R)^{-1}\mathbf{y}_{l,4},\ 1 \le l \le 2,$$

where

$$\mathbf{y}_{1,1} = \mathbf{0}, \mathbf{y}_{1,2} = \sum_{k=0}^{2} \mathbf{e}_{2k}(6), \mathbf{y}_{1,3} = \mathbf{e}_0(5) + \sum_{k=0}^{1} \mathbf{e}_{2k+1}(5), \mathbf{y}_{1,4} = \sum_{k=0}^{1} \mathbf{e}_{2k}(4),$$

$$\mathbf{y}_{2,1} = \mathbf{e}_1(3), \mathbf{y}_{2,2} = \sum_{k=2}^{3} \mathbf{e}_k(6), \mathbf{y}_{2,3} = \sum_{k=1}^{2} \mathbf{e}_k(5), \mathbf{y}_{2,4} = \sum_{k=0}^{1} \mathbf{e}_k(4).$$

Let us denote by T_n, $1 \le n \le 4$, the respective random times to first failure of the system (failure of both servers), or of one server (either server 1 or server 2). The corresponding reliability function, which is the same as the complementary cumulative distribution function of the lifetime T_n, is then defined as

$$R_n(t) = \mathbb{P}[T_n > t], 1 \le n \le 4.$$

In this section, we obtain these functions in terms of the Laplace transform $\tilde{R}_n(s) = \int_0^\infty R(t)e^{-st}dt, Re[s] > 0$. To this end, we let the corresponding failure states be absorbing states. We thus obtain new processes that can be modelled by auxiliary continuous-time absorbing Markov chains $\{\hat{X}_m(t)\}_{t\ge 0}$ with state spaces $\hat{E}_n, 1 \le n \le 4$, where $\hat{E}_1 = E \setminus \{x = (q, 2, 2); q \in \mathbb{N}_0\}, \hat{E}_2 = E \setminus \{x = (q, d_1, d_2); q \in \mathbb{N}_0, d_1 = 2 \vee d_2 = 2\}, \hat{E}_3 = E \setminus \{x = (q, 2, d_2); q \in \mathbb{N}_0, d_2 \in \{0, 1, 2\}\}$ and $\hat{E}_4 = E \setminus \{x = (q, d_1, 2); q \in \mathbb{N}_0, d_1 \in \{0, 1, 2\}\}$. Two approaches can be used to obtain the function $\tilde{R}_n(s)$: (i) a classical method based on transient solution of the auxiliary absorbing Markov chain and (ii) an alternative method which calculates the distribution of the first passage time to the absorbing state using the conditional remaining life time distributions. These methods contain two main steps including evaluation of the Laplace–Stiltjes transforms of the state probabilities of the absorbing Markov chain or of the remaining life time given initial state and subsequent derivation of generating functions of the corresponding transforms. Additionally to the reliability function we will analyse in the paper the discrete counterpart in form of distribution of the number of failures during a life time. Therefore, the second method seems to be more preferable and logically suitable for the proposed reliability analysis framework. The description of the first method is given only for the function $\tilde{R}_2(s)$ (see below), while the second one is implemented for all functions.

Theorem 2. *The Laplace transform of $R_2(t)$ is given by*

$$\tilde{R}_2(s) = \tilde{P}_{1,0}(s,1) + \tilde{P}_{1,1}(s,1) + \tilde{P}_{1,2}(s,1), \tag{10}$$

where

$$\tilde{P}_{1,0}(s,1) = \frac{1 + \alpha_1\tilde{\pi}_{(0,0,0)}(s) - \lambda\tilde{\pi}_{(q_1-1,1,0)}(s) + \mu_2\tilde{P}_{1,1}(s,1)}{s+\alpha_1}, \tag{11}$$

$$\tilde{P}_{1,1}(s,1)=\frac{\alpha_1\tilde{\pi}_{(0,0,1)}(s)+\lambda(\tilde{\pi}_{(q_1-1,1,0)}(s)-\tilde{\pi}_{(q_1-1,1,1)}(s))+\mu\tilde{\pi}_{(q_1,1,1)}(s)}{s+\alpha+\mu_2},$$

$$\tilde{P}_{1,2}(s,1)=\frac{\lambda\tilde{\pi}_{(q_1-1,1,1)}(s)-\mu\tilde{\pi}_{(q_1,1,1)}(s)}{s+\alpha},$$

the functions $\tilde{\pi}_x(s)$ are of the form,

$$\tilde{\pi}_{(q_1,1,1)}(s)=\frac{\lambda z(s)\tilde{L}_{q_1}(s)\mathbf{e}_1(2)}{\mu-\lambda z(s)\tilde{M}_{q_1}(s)\mathbf{e}_1(2)}, \tag{12}$$

$$(\tilde{\pi}_{(q_1-1,1,0)}(s),\tilde{\pi}_{(q_1-1,1,1)}(s))=\tilde{\pi}_{(q_1,1,1)}(s)\tilde{M}_{q_1}(s)+\tilde{L}_{q_1}(s), \tag{13}$$

$$(\tilde{\pi}_{(0,0,0)}(s),\tilde{\pi}_{(0,0,1)}(s))=\tilde{\pi}_{(q_1,1,1)}(s)\prod_{i=0}^{q_1}\tilde{M}_{q_1-i}(s)+\sum_{i=0}^{q_1}\tilde{L}_{q_1-i}(s)\prod_{j=i+1}^{q_1}\tilde{M}_{q_1-j}(s), \tag{14}$$

the matrices $\tilde{M}_i(s)$ and $\tilde{L}_i(s)$ are evaluated recursively,

$$\begin{aligned}&\tilde{M}_0(s)=\mu_1\tilde{N}_0(s),\tilde{L}_0(s)=\mathbf{e}_0'(2)\tilde{N}_0(s),\tilde{N}_0(s)=-(\hat{Q}_{1,0}-sI_2)^{-1},\\&\tilde{M}_q(s)=\mu_1\tilde{N}_q(s),\tilde{L}_q(s)=\lambda\tilde{L}_{q-1}(s)\tilde{N}_q(s),\tilde{N}_q(s)=-(\hat{Q}_{1,1}-sI_2+\lambda\tilde{M}_{q-1}(s))^{-1}q=\overline{1,q_1-1},\\&\tilde{M}_{q_1}(s)=-\mu\mathbf{e}_1'(2)\tilde{N}_{q_1}(s),\tilde{L}_{q_1}(s)=-\lambda\tilde{L}_{q_1-1}\tilde{N}_{q_1}(s),\tilde{N}_{q_1}(s)=(\hat{Q}_{1,2}-sI_2+\lambda\tilde{M}_{q_1-1}(s))^{-1},\end{aligned} \tag{15}$$

the matrices $\hat{Q}_{1,0},\hat{Q}_{1,1}$ and $\hat{Q}_{1,2}$ are of the form

$$\hat{Q}_{1,0}=\begin{pmatrix}-\lambda&0\\\mu_2&-(\lambda+\alpha_2+\mu_2)\end{pmatrix},\hat{Q}_{1,1}=\begin{pmatrix}-(\lambda+\alpha_1+\mu_1)&0\\\mu_2&-(\lambda+\alpha+\mu)\end{pmatrix},\hat{Q}_{1,2}=\begin{pmatrix}-(\lambda+\alpha_1+\mu_1)&\lambda\\\mu_2&-(\lambda+\alpha+\mu)\end{pmatrix},$$

and the function $z(s)$ is defined as

$$z(s)=\frac{s+\alpha+\lambda+\mu}{2\lambda}-\sqrt{\left(\frac{s+\alpha+\lambda+\mu}{2\lambda}\right)^2-\frac{\mu}{\lambda}}. \tag{16}$$

Proof. The absorbing states of the process $\{\hat{X}_2(t)\}$ are $x=(q,2,d_2),d_2\in\{0,1,2\}$ and $x=(q,d_1,2),d_1\in\{0,1,2\}$. Using the same notations as in the previous section, we obtain the following set of Kolmogorov differential equations:

$$\begin{aligned}&\pi'_{(0,0,0)}(t)=-\lambda\pi_{(0,0,0)}(t)+\mu_1\pi_{(0,1,0)}(t)+\mu_2\pi_{(0,0,1)}(t),\\&\pi'_{(q,1,0)}(t)=-(\alpha_1+\lambda+\mu_1)\pi_{(q,1,0)}+\lambda\pi_{(q-1,1,0)}(t)+\mu_1\pi_{(q+1,1,0)}(t)+\mu_2\pi_{(q,1,1)}(t),0\le q\le q_1-2,\\&\pi'_{(q_1-1,1,0)}(t)=-(\alpha_1+\lambda+\mu_1)\pi_{(q_1-1,1,0)}+\lambda\pi_{(q_1-2,1,0)}(t)+\mu_2\pi_{(q_1-1,1,1)},\\&\pi'_{(0,0,1)}(t)=-(\alpha_2+\lambda+\mu_2)\pi_{(0,0,1)}(t)+\mu_1\pi_{(0,1,1)}(t),\\&\pi'_{(0,1,1)}(t)=-(\alpha_2+\lambda+\mu_2)\pi_{(0,0,1)}(t)+\lambda\pi_{(0,0,1)}(t)+\mu_1\pi_{(0,1,1)}(t),\\&\pi'_{(q,1,1)}(t)=-(\alpha+\lambda+\mu)\pi_{(q,1,1)}(t)+\lambda\pi_{(q-1,1,1)}(t)+\mu_1\pi_{(q+1,1,1)}(t),1\le q\le q_1-2,\\&\pi'_{(q_1-1,1,1)}(t)=-(\alpha+\lambda+\mu)\pi_{(q_1-1,1,1)}(t)+\lambda\pi_{(q_1-1,1,0)}(t)+\lambda\pi_{(q_1-2,1,1)}(t)+\mu\pi_{(q_1,1,1)}(t)\end{aligned} \tag{17}$$

with initial conditions $\pi_{(0,0,0)}(0)=1$ and $\pi_x(0)=0,x\in\hat{E}_2$. By taking Laplace transforms of these equations, where $\tilde{\pi}_x(s)=\int_0^\infty\pi_x(t)e^{-st}dt,Re[s]\ge 0$, and then using their partial generating functions,

$$\tilde{P}_{1,0}(s,z)=\tilde{\pi}_{(0,0,0)}(s)+\sum_{q=0}^{q_1-1}\tilde{\pi}_{(q,1,0)}(s)z^{i+1},\tilde{P}_{1,1}(s,z)=\tilde{\pi}_{(0,0,1)}(s)+\sum_{q=0}^{q_1-1}\tilde{\pi}_{(q,1,1)}(s)z^{i+1},$$

$$\tilde{P}_{1,2}(s,z)=\sum_{q=q_1}^{\infty}\tilde{\pi}_{(q,1,1)}(s)z^{i+1}$$

for $|z|<1$, the system (17) is transformed after some manipulation into a set of equations for the double transforms introduced above:

$$\tilde{P}_{1,0}(s,z)=\frac{z+\tilde{\pi}_{(0,0,0)}(s)(\mu_1(z-1)+\alpha_1 z)-\lambda z^{q_1+2}\tilde{\pi}_{(q_1-1,1,0)}(s)+\mu_2 z\tilde{P}_{1,1}(s,z)}{-\lambda z^2+(s+\alpha_1+\lambda+\mu_1)z-\mu_1},$$

$$\tilde{P}_{1,1}(s,z)=\frac{\tilde{\pi}_{(0,0,0)}(s)(z(\alpha_1+\mu_1)-\mu_1)+\lambda(\tilde{\pi}_{(q_1-1,1,0)}(s)-z^{q_1+1}\pi_{(q_1-1,1,1)}(s))+\mu\tilde{\pi}_{(q_1,1,1)}(s)}{-\lambda z^2+(s+\alpha+\lambda+\mu)z-\mu_1},$$

$$\tilde{P}_{1,2}(s,z)=\frac{z^{q_1+1}(\lambda z\tilde{\pi}_{(q_1-1,1,1)}(s)-\mu\tilde{\pi}_{(q_1,1,1)}(s))}{-\lambda z^2+(s+\alpha+\lambda+\mu)z-\mu}.$$

Let us denote by $F(s,z)=-\lambda z^2+(s+\alpha+\lambda+\mu)z-\mu$ the auxiliary function for the denominator of $\tilde{P}_{1,2}(s,z)$. It is easy to see that

$F(s,0)=-\mu<0,\quad F(s,1)=s+\alpha\ge 0.$

Thus, for any $s > 0$ the square equation $F(s,z) = 0$ has two roots and their minimum takes the value in the interval [0,1]. This root we denote as

$$z(s) = \frac{s+\alpha+\lambda+\mu}{2\lambda} - \sqrt{\left(\frac{s+\alpha+\lambda+\mu}{2\lambda}\right)^2 - \frac{\mu}{\lambda}}.$$

Since the function $\tilde{P}_{1,2}(s,z)$ is analytical, its numerator must also be zero at point $z = z(s)$:

$$\lambda z(s)\tilde{\pi}_{(q_1-1,1,1)}(s) - \mu\tilde{\pi}_{(q_1,1,1)}(s) = 0. \tag{18}$$

In order to obtain a second equation for the boundary transforms $\tilde{\pi}_{(q_1-1,1,1)}(s)$ and $\tilde{\pi}_{(q_1,1,1)}(s)$, let us denote by

$$\tilde{\boldsymbol{\pi}}_{0,0}(s) = (\tilde{\pi}_{(0,0,0)}(s), \tilde{\pi}_{(0,0,1)}(s)), \tilde{\boldsymbol{\pi}}_{q,1}(s) = (\tilde{\pi}_{(q,1,0)}(s), \tilde{\pi}_{(q,1,1)}(s)), 1 \le q \le q_1 - 1.$$

For the system of the Laplace transforms $\tilde{\pi}_x(s)$ obtained from (17), we can use the following relations in matrix form:

$$\tilde{\boldsymbol{\pi}}_{0,0}(s) = -\mu_1\tilde{\boldsymbol{\pi}}_{0,1}(s)(\hat{Q}_{1,0} - sI_2)^{-1} - \mathbf{e}'_0(2)(\hat{Q}_{1,0} - sI_2)^{-1} = \tilde{\boldsymbol{\pi}}_{0,1}(s)\tilde{M}_0(s) + \tilde{L}_0(s).$$

Substituting the last expression into the matrix relation for $\tilde{\pi}_{0,1}(s)$ yields

$$\tilde{\boldsymbol{\pi}}_{0,1}(s) = -\mu_1\tilde{\boldsymbol{\pi}}_{1,1}(s)(\hat{Q}_{1,1} - sI_2 + \lambda\tilde{M}_0(s))^{-1} - \lambda\tilde{L}_0(s)(\hat{Q}_{1,1} - sI_2 + \lambda\tilde{M}_0(s))^{-1} = \tilde{\boldsymbol{\pi}}_{1,1}(s)\tilde{M}_1(s) + \tilde{L}_1(s).$$

Sequential application of this forward-elimination-backward-substitution method leads to the following recursive relations:

$$\tilde{\boldsymbol{\pi}}_{q-1,1}(s) = \tilde{\boldsymbol{\pi}}_{q,1}(s)\tilde{M}_q(s) + \tilde{L}_q(s), 1 \le q \le q_1 - 2,$$
$$\tilde{\boldsymbol{\pi}}_{q_1-1,1}(s) = \tilde{\pi}_{(q_1,1,1)}(s)\tilde{M}_{q_1}(s) + \tilde{L}_{q_1}(s),$$

where $\tilde{M}_q(s)$ and $\tilde{L}_q(s)$ can be calculated by (15). By combining the relation

$$\tilde{\pi}_{(q_1-1,1,1)}(s) = (\pi_{q_1,1,1}(s)\tilde{M}_{q_1}(s) + \tilde{L}_{q_1}(s))\mathbf{e}_1(2)$$

and (18), we can express $\tilde{\pi}_{(q_1,1,1)}(s)$ in the form (12). The transforms for the remaining boundary states can thus be evaluated as functions of $\tilde{\pi}_{(q_1-1,1,1)}(s)$. Finally, the double transforms are calculated at point $z = 1$ and substituted into (10). □

For evaluating the reliability function, the next four statements use the second approach based on calculating the LSTs of the probability density function of the remaining life time.

Theorem 3. *The Laplace transform of $R_1(t)$ is given by*

$$\tilde{R}_1(s) = \frac{1}{s}\left[1 - \mathbf{e}'_0(3)\prod_{j=0}^{q_1}\tilde{M}_j(s)\tilde{\mathbf{r}}_{q_1,1}(s) - \mathbf{e}'_0(3)\sum_{i=1}^{q_1}\prod_{j=0}^{i-1}\tilde{M}_j(s)\tilde{L}_i(s)\right], \tag{19}$$

where $\tilde{M}_i(s), 0 \le i \le q_1$, and $\tilde{L}_i(s), 1 \le i \le q_1$, are evaluated recursively,

$$\tilde{M}_0(s) = \tilde{N}_0(s)\hat{Q}_{0,1}, \tilde{N}_0(s) = -(\hat{Q}_{1,0} - sI_3)^{-1}, \tag{20}$$
$$\tilde{M}_1(s) = \lambda\tilde{N}_1(s), \tilde{L}_1(s) = \tilde{N}_1(s)(\alpha_1\mathbf{e}_4(5) + \alpha_2\mathbf{e}_3(5)), \tilde{N}_1(s) = -(\hat{Q}_{1,1} - sI_5 + \hat{Q}_{2,1}\tilde{M}_0(s))^{-1},$$
$$\tilde{M}_q(s) = \lambda\tilde{N}_q(s), \tilde{L}_q(s) = \tilde{N}_q(s)(\hat{Q}_{2,2}\tilde{L}_{q-1}(s) + \alpha_1\mathbf{e}_4(5) + \alpha_2\mathbf{e}_3(5)), 2 \le q \le q_2 - 1,$$
$$\tilde{N}_q(s) = -(\hat{Q}_{1,1} - sI_5 + \hat{Q}_{2,2}\tilde{M}_{q-1}(s))^{-1}, 2 \le q \le q_2 - 1,$$
$$\tilde{M}_{q_2}(s) = \tilde{N}_{q_2}(s)\hat{Q}_{0,2}, \tilde{L}_{q_2}(s) = \tilde{N}_{q_2-1}(s)(\hat{Q}_{2,2}\tilde{L}_{q_2-1}(s) + \alpha_1\mathbf{e}_4(5) + \alpha_2\mathbf{e}_3(5)),$$
$$\tilde{N}_{q_2}(s) = -(\hat{Q}_{1,2} - sI_5 + \hat{Q}_{2,2}\tilde{M}_{q_2-1}(s))^{-1},$$
$$\tilde{M}_{q_2+1}(s) = \lambda\tilde{N}_{q_2+1}(s), \tilde{L}_{q_2+1}(s) = \tilde{N}_{q_2+1}(s)(\hat{Q}_{2,3}\tilde{L}_{q_2}(s) + \alpha_1\mathbf{e}_3(4) + \alpha_2\mathbf{e}_2(4)),$$
$$\tilde{N}_{q_2+1}(s) = -(\hat{Q}_{1,3} - sI_4 + \hat{Q}_{2,3}\tilde{M}_{q_2}(s))^{-1},$$
$$\tilde{M}_q(s) = \lambda\tilde{N}_q(s), \tilde{L}_q(s) = \tilde{N}_q(s)(\hat{Q}_{2,4}\tilde{L}_{q-1}(s) + \alpha_1\mathbf{e}_3(4) + \alpha_2\mathbf{e}_2(4)),$$
$$\tilde{N}_q(s) = -(\hat{Q}_{1,3} - sI_4 + \hat{Q}_{2,4}\tilde{M}_{q-1}(s))^{-1}, q_2 + 2 \le q \le q_1 - 1,$$
$$\tilde{M}_{q_1}(s) = \tilde{N}_{q_1}(s)\hat{Q}_{0,3}, \tilde{L}_{q_1}(s) = \tilde{N}_{q_1}(s)(\hat{Q}_{2,4}\tilde{L}_{q_1-1}(s) + \alpha_1\mathbf{e}_3(4) + \alpha_2\mathbf{e}_2(4)),$$
$$\tilde{N}_{q_1}(s) = -(\hat{Q}_{1,4} - sI_4 + \hat{Q}_{2,4}\tilde{M}_{q_1-1}(s))^{-1}.$$

The matrices $\hat{Q}_{1,i}, 0 \le i \le 4$, $\hat{Q}_{0,i}, 1 \le i \le 3$, and $\hat{Q}_{2,i}, 1 \le i \le 4$, are of the form

$$\hat{Q}_{1,0} = \begin{pmatrix} -\lambda & 0 & 0 \\ \mu_2 & -(\alpha_2+\lambda+\mu_2) & \alpha_2 \\ 0 & \beta_2 & -(\beta_2+\lambda) \end{pmatrix},$$

$$\hat{Q}_{1,1} = \begin{pmatrix} -(\alpha_1+\lambda+\mu_1) & \alpha_1 & 0 & 0 & 0 \\ \beta_1 & -(\beta_1+\lambda) & 0 & 0 & 0 \\ \mu_2 & 0 & -(\alpha+\lambda+\mu) & \alpha_1 & \alpha_2 \\ 0 & \mu_2 & \beta_1 & -(\alpha_2+\beta_1+\lambda+\mu_2) & 0 \\ 0 & 0 & \beta_2 & 0 & -(\alpha_1+\beta_2+\lambda+\mu_1) \end{pmatrix},$$

$$\hat{Q}_{1,2} = \hat{Q}_{1,1} + \lambda \mathbf{e}_1(5) \otimes \mathbf{e}'_3(5), \hat{Q}_{1,3} = \begin{pmatrix} -(\alpha_1+\lambda+\mu_1) & 0 & 0 & 0 \\ \mu_2 & -(\alpha+\lambda+\mu) & \alpha_1 & \alpha_2 \\ 0 & \beta_1 & -(\alpha_2+\beta_1+\lambda+\mu_2) & 0 \\ 0 & \beta_2 & 0 & -(\alpha_1+\beta_2+\lambda+\mu_1) \end{pmatrix},$$

$$\hat{Q}_{1,4} = \hat{Q}_{1,3} + \lambda \mathbf{e}_0(4) \otimes \mathbf{e}'_1(4), \hat{Q}_{0,1} = \lambda \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \hat{Q}_{0,2} = \lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \hat{Q}_{0,3} = \lambda \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\hat{Q}_{2,1} = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mu_1 \end{pmatrix}, \hat{Q}_{2,2} = \begin{pmatrix} \mu_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_1 \end{pmatrix}, \hat{Q}_{2,3} = \begin{pmatrix} \mu_1 & 0 & 0 & \alpha_1 & 0 \\ 0 & 0 & \mu_1 & 0 & 0 \\ 0 & 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & 0 & \mu_1 \end{pmatrix}, \hat{Q}_{2,4} = \begin{pmatrix} \mu_1 & 0 & \alpha_1 & 0 \\ 0 & \mu_1 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & \mu_1 \end{pmatrix}.$$

The column vector $\tilde{\mathbf{r}}_{q_1,1}(s) = (\tilde{r}_{(q_1,1,1)}(s), \tilde{r}_{(q_1,2,1)}(s), \tilde{r}_{(q_1,1,2)}(s))'$ *is a solution of the system*

$$\tilde{\mathbf{r}}_{q_1-1,1}(s) = \tilde{M}_{q_1}(s)\tilde{\mathbf{r}}_{q_1,1}(s) + \tilde{L}_{q_1}(s), \tag{21}$$

$$X_3(z)X_4(z)Y_1(s,z) + X_3(z)Z_1(s,z)[X_4(z) + Y_2(s,z)] + Y_1(s,z)[X_2(s,z)Z_1(s,z) + X_4(z)Z_2(s,z)]\Big|_{z=z(s)} = 0, \tag{22}$$

$$\begin{aligned} &X_3(z)X_4(z)Y_3(z) + X_1(s,z)Z_1(s,z)[X_4(z) + Y_2(s,z)] + Y_3(z)[X_2(s,z)Z_1(s,z) + X_4(z)Z_2(s,z)] \\ &\quad - X_4(z)Z_3(z)[X_4(z) + Y_2(s,z)]\Big|_{z=z(s)} = 0, \end{aligned} \tag{23}$$

$$\begin{aligned} &X_2(s,z)Y_1(s,z)Z_3(z) + [X_3(z) + Z_2(s,z)][X_1(s,z)Y_1(s,z) - X_3(z)Y_3(z)] \\ &\quad + X_3(z)Z_3(z)[X_4(z) + Y_2(s,z)]\Big|_{z=z(s)} = 0, \end{aligned} \tag{24}$$

where

$$\begin{aligned} &X_1(s,z) = (s+\alpha+\lambda+\mu)z - \lambda - \mu z^2, \\ &Y_1(s,z) = (1-z)[(s+\alpha_2+\beta_1+\lambda+\mu_2)z - \lambda - \mu_2 z^2], \\ &Z_1(s,z) = (1-z)[(s+\alpha_1+\beta_2+\lambda+\mu_1)z - \lambda - \mu_1 z^2], \\ &X_2(s,z) = \mu z \tilde{r}_{(q_1-1,1,1)}(s) - \lambda \tilde{r}_{(q_1,1,1)}(s), \\ &Y_2(s,z) = \mu_2 z(1-z)\tilde{r}_{(q_1-1,2,1)}(s) - \lambda(1-z)\tilde{r}_{(q_1,2,1)}(s), \\ &Z_2(s,z) = \mu_1 z(1-z)\tilde{r}_{(q_1-1,1,2)}(s) - \lambda(1-z)\tilde{r}_{(q_1,1,2)}(s), \\ &X_3(z) = \alpha_1 z, X_4(z) = \alpha_2 z, Y_3(z) = \beta_1 z(1-z), Z_3(z) = \beta_2 z(1-z), \end{aligned} \tag{25}$$

and the function $z(s)$ *is a minimal solution in the interval [0,1] of the equation*

$$X_1(s,z)Y_1(s,z)Z_1(s,z) - X_3(z)Y_3(z)Z_1(s,z) - X_4(z)Y_1(s,z)Z_3(z) = 0. \tag{26}$$

Proof. First we introduce some notation:

T_x – the first passage time to the absorbing state of the process $\{\hat{X}_1(t)\}$ given that the initial state is $x \in \hat{E}_1$;
$r_x(t) = \frac{1}{dt}\mathbb{P}[T_x \in [x, x+dx)]$ – the probability density function (PDF) of T_x;
$\tilde{r}_x(s) = \int_0^\infty r_x(t)e^{-st}dt, \mathrm{Re}[s] > 0$ – the Laplace–Stiltjes transform (LST) of $r_x(t)$.

According to the first-step analysis, the LSTs $\tilde{r}_x(s)$ satisfy

$$\tilde{r}_x(s) = \sum_{y \neq x} \frac{\lambda_{xy}}{s+\lambda_x}\tilde{r}_y(s), x \in \hat{E}_1, \tag{27}$$

where $\lambda_x = \sum_{y \neq x} \lambda_{xy}$. Consider now the states x with a queue length $q \geq q_1$. We employ first-step analysis to obtain the system

$$\begin{aligned} &-(s+\alpha+\lambda+\mu)\tilde{r}_{(q,1,1)}(s) + \alpha_1 \tilde{r}_{(q,2,1)}(s) + \alpha_2 \tilde{r}_{(q,1,2)}(s) + \lambda \tilde{r}_{(q+1,1,1)}(s) + \mu \tilde{r}_{(q-1,1,1)}(s) = 0, \\ &-(s+\alpha_2+\beta_1+\lambda+\mu_2)\tilde{r}_{(q,2,1)}(s) = \alpha_2 + \beta_1 \tilde{r}_{(q,1,1)}(s) + \lambda \tilde{r}_{(q+1,2,1)}(s) + \mu_2 \tilde{r}_{(q-1,2,1)}(s) = 0, \\ &-(s+\alpha_1+\beta_2+\lambda+\mu_1)\tilde{r}_{(q,1,2)}(s) = \alpha_1 + \beta_2 \tilde{r}_{(q,1,1)}(s) + \lambda \tilde{r}_{(q+1,1,2)}(s) + \mu_1 \tilde{r}_{(q-1,1,2)}(s) = 0. \end{aligned} \tag{28}$$

Let us define the following partial generating functions:

$$\tilde{P}_{1,1}(s,z) = \sum_{q=q_1}^{\infty} \tilde{r}_{(q,1,1)}(s)z^{q-q_1}, \tilde{P}_{2,1}(s,z) = \sum_{q=q_1}^{\infty} \tilde{r}_{(q,2,1)}(s)z^{q-q_1}, \tilde{P}_{1,2}(s,z) = \sum_{q=q_1}^{\infty} \tilde{r}_{(q,1,2)}(s)z^{q-q_1}, |z| < 1. \tag{29}$$

Multiplying Eq. (28) by z^q and summing over q, we obtain the following equations after some algebraic manipulation:

$$\tilde{P}_{1,1}(s,z)=\frac{X_3(z)X_4(z)Y_1(s,z)+X_3(z)Z_1(s,z)[X_4(z)+Y_2(s,z)]+Y_1(s,z)[X_2(s,z)Z_1(s,z)+X_4(z)Z_2(s,z)]}{X_1(s,z)Y_1(s,z)Z_1(s,z)-X_3(z)Y_3(z)Z_1(s,z)-X_4(z)Y_1(s,z)Z_3(z)}, \tag{30}$$

$$\tilde{P}_{2,1}(s,z)=\frac{X_3(z)X_4(z)Y_3(z)+X_1(s,z)Z_1(s,z)[X_4(z)+Y_2(s,z)]+Y_3(z)[X_2(s,z)Z_1(s,z)+X_4(z)Z_2(s,z)]-X_4(z)Z_3(z)[X_4(z)+Y_2(s,z)]}{X_1(s,z)Y_1(s,z)Z_1(s,z)-X_3(z)Y_3(z)Z_1(s,z)-X_4(z)Y_1(s,z)Z_3(z)}, \tag{31}$$

$$\tilde{P}_{1,2}(s,z)=\frac{X_2(s,z)Y_1(s,z)Z_3(z)+[X_3(z)+Z_2(s,z)][X_1(s,z)Y_1(s,z)-X_3(z)Y_3(z)]+X_3(z)Z_3(z)[X_4(z)+Y_2(s,z)]}{X_1(s,z)Y_1(s,z)Z_1(s,z)-X_3(z)Y_3(z)Z_1(s,z)-X_4(z)Y_1(s,z)Z_3(z)}, \tag{32}$$

where the involved functions $X_i(s,z)$, $Y_i(s,z)$, $Z_i(s,z)$, $1\le i\le 3$, $X_3(z)$, $Y_3(z)$, $Z_3(z)$ and $X_4(z)$ are defined as in (25). Consider the denominator $F(s,z)=X_1(s,z)Y_1(s,z)Z_1(s,z)-X_3(z)Y_3(z)Z_1(s,z)-X_4(z)Y_1(s,z)Z_3(z)$. It is clear that

$$F(s,0)=-\lambda^3<0,\ F(s,1)=\alpha_1^2(s+\alpha_2)+s(s+\alpha_2+\beta_1)(s+\alpha_2+\beta_2)+\alpha_1(s+\alpha_2)(2s+\alpha_2+\beta)\ge 0.$$

Thus, $F(s,z)$ for any s has at least one root $z(s)$ in the interval [0,1]. Since the functions $\tilde{P}_{1,1}(s,z),\tilde{P}_{2,1}(s,z)$ and $\tilde{P}_{2,1}(s,z)$ are analytical, the numerators in (30)–(32) must also be zero at point $z(s)$. This leads to the system with three Eqs. (22)–(24) with six unknown boundary functions $\tilde{r}_{(q,1,1)}(s),\tilde{r}_{(q,2,1)}(s)$ and $\tilde{r}_{(q,1,2)}(s)$ for $q\in\{q_1-1,q_1\}$. To obtain another three equations for the specified boundary functions, consider the states of $\{\hat{X}_1(t)\}_{t\ge 0}$ below threshold level q_1 and define the following vectors, which comprise the LSTs $\tilde{r}_x(s)$:

$$\tilde{\mathbf{r}}_{0,0}(s)=(\tilde{r}_{(0,0,0)}(s),\tilde{r}_{(0,0,1)}(s),\tilde{r}_{(0,0,2)}(s))', \tag{33}$$

$$\tilde{\mathbf{r}}_{q,1}(s)=\begin{cases}(\tilde{r}_{(q,1,0)}(s),\tilde{r}_{(q,2,0)}(s),\tilde{r}_{(q,1,1)}(s),\tilde{r}_{(q,2,1)}(s),\tilde{r}_{(q,1,2)}(s))', & 0\le q\le q_2-1,\\ (\tilde{r}_{(q,1,0)}(s),\tilde{r}_{(q,1,1)}(s),\tilde{r}_{(q,2,1)}(s),\tilde{r}_{(q,1,2)}(s))', & q_2\le q\le q_1-1,\\ (\tilde{r}_{(q,1,1)}(s),\tilde{r}_{(q,2,1)}(s),\tilde{r}_{(q,1,2)}(s))', & q=q_1.\end{cases}$$

Further, we have $\tilde{r}_{(q,2,2)}(s)=1$ for any $q\in\mathbb{N}_0$. By expressing Eq. (27) in matrix form and solving this system using the forward-elimination-backward-substitution method we obtain the following recursive relations,

$$\tilde{\mathbf{r}}_{0,0}=\tilde{M}_0(s)\tilde{\mathbf{r}}_{0,1}(s), \tag{34}$$

$$\tilde{\mathbf{r}}_{q,1}=\tilde{M}_{q+1}(s)\tilde{\mathbf{r}}_{q+1,1}(s)+\tilde{L}_{q+1}(s),\ 0\le q\le q_1-1,$$

where $\tilde{M}_i(s), 0\le i\le q_1$, and $\tilde{L}_i(s), 1\le i\le q_1$, can be calculated by (20). Particularly, we obtain the expression

$$\tilde{\mathbf{r}}_{q_1-1,1}=\tilde{M}_{q_1}(s)\tilde{\mathbf{r}}_{q_1,1}(s)+\tilde{L}_{q_1}(s),$$

which gives us the missing three equations for the boundary functions. Hence, we have the system (21)–(24) of six equations with six unknowns, which can be solved. Note that the Laplace transform $\tilde{R}_1(s)$ can be represented as a function of $\tilde{r}_{(0,0,0)}(s)$ as

$$\tilde{R}_1(s)=\frac{1}{s}\Big[1-\tilde{r}_{(0,0,0)}(s)\Big]. \tag{35}$$

Finally, the recursive substitution in (34) gives the relation

$$\tilde{\mathbf{r}}_{0,0}(s)=\prod_{j=0}^{q_1}\tilde{M}_j(s)\tilde{r}_{(q_1,1,1)}(s)+\sum_{i=1}^{q_1}\prod_{j=0}^{i-1}\tilde{M}_j(s)\tilde{L}_i(s), \tag{36}$$

which together with $\tilde{r}_{(0,0,0)}(s)=\mathbf{e}_0'(3)\tilde{\mathbf{r}}_{0,0}(s)$ and (35) completes the proof. □

Remark 1. Solving the sextic Eq. (26) in symbolic form is difficult. However, a solution may be found by approximating the infinite buffer model using a finite buffer with a sufficiently large buffer truncation parameter B. In this case, (19) can be rewritten in the form

$$\tilde{R}_1(s)\approx\frac{1}{s}\left[1-\mathbf{e}_0'(3)\sum_{i=1}^{B+1}\prod_{j=0}^{i-1}\tilde{M}_j(s)\tilde{L}_i(s)\right],$$

where

$$\tilde{M}_{q_1+1}(s)=\lambda\tilde{N}_{q_1+1}(s),\ \tilde{L}_{q_1+1}(s)=\tilde{N}_{q_1+1}(s)(\hat{Q}_{2,5}\tilde{L}_{q_1}(s)+\alpha_1\mathbf{e}_2(3)+\alpha_2\mathbf{e}_1(3)),$$

$$\tilde{N}_{q_1+1}(s)=(\hat{Q}_{1,5}-sI_3+\hat{Q}_{2,5}\tilde{M}_{q_1}(s))^{-1},$$

$$\tilde{M}_q(s)=\lambda\tilde{N}_{q_1+1}(s),\ \tilde{L}_q(s)=\tilde{N}_q(s)(\hat{Q}_{2,6}\tilde{L}_{q-1}(s)+\alpha_1\mathbf{e}_2(3)+\alpha_2\mathbf{e}_1(3)),\ q_1+2\le q\le B-1,$$

$$\tilde{N}_q(s)=(\hat{Q}_{1,5}-sI_3+\hat{Q}_{2,6}\tilde{M}_{q-1}(s))^{-1},\ q_1+2\le q\le B-1,$$

$$\tilde{L}_B(s)=\tilde{N}_B(s)(\hat{Q}_{2,6}\tilde{L}_{B-1}(s)+\alpha_1\mathbf{e}_2(3)+\alpha_2\mathbf{e}_1(3)),\ \tilde{N}_B(s)=(\hat{Q}_{1,6}-sI_3+\hat{Q}_{2,6}\tilde{M}_{B-1}(s))^{-1}$$

and

$$\hat{Q}_{1,5} = \begin{pmatrix} -(\alpha+\lambda+\mu) & \alpha_1 & \alpha_2 \\ \beta_1 & -(\alpha_2+\beta_1+\lambda+\mu_2) & 0 \\ \beta_2 & 0 & -(\alpha_1+\beta_2+\lambda+\mu_1) \end{pmatrix}, \hat{Q}_{1,6} = \hat{Q}_{1,5} + \lambda I_3,$$

$$\hat{Q}_{2,5} = \begin{pmatrix} 0 & \mu & 0 & 0 \\ 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & \mu_1 \end{pmatrix}, \hat{Q}_{2,6} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_1 \end{pmatrix}.$$

Theorem 4. *The Laplace transform of $R_2(t)$ is given by*

$$\tilde{R}_2(s) = \frac{1}{s}\left[1 - \tilde{r}_{(q_1,1,1)}(s)\mathbf{e}_0'(2)\prod_{j=0}^{q_1}\tilde{M}_j(s) - \mathbf{e}_0'(2)\sum_{i=0}^{q_1}\prod_{j=0}^{i-1}\tilde{M}_j(s)\tilde{L}_i(s)\right], \tag{37}$$

where $\tilde{M}_i(s), 0 \le i \le q_1$, and $\tilde{L}_i(s), 1 \le i \le q_1$, are evaluated recursively,

$$\begin{aligned} &\tilde{M}_0(s) = \lambda\tilde{N}_0(s), \tilde{L}_0(s) = \alpha_2 N_0(s)\mathbf{e}_1(2), \tilde{N}_0(s) = -(\hat{Q}_{1,0} - sI_2)^{-1}, \\ &\tilde{M}_q(s) = \lambda\tilde{N}_q(s), \tilde{L}_q(s) = \tilde{N}_q(s)(\mu_1\tilde{L}_{q-1}(s) + \alpha_1\mathbf{e}(2) + \alpha_2\mathbf{e}_1(2)), 1 \le q \le q_1 - 1, \\ &\tilde{N}_q(s) = -(\hat{Q}_{1,1} - sI_2 + \mu_1\tilde{M}_{q-1}(s))^{-1}, 1 \le q \le q_1 - 1, \\ &\tilde{M}_{q_1}(s) = \lambda\tilde{N}_{q_1}(s), \tilde{L}_{q_1}(s) = \tilde{N}_{q_1}(s)(\mu_1\tilde{L}_{q_1-1}(s) + \alpha_1\mathbf{e}(2) + \alpha_2\mathbf{e}_1(2)), \\ &\tilde{N}_{q_1}(s) = -(\hat{Q}_{1,2} - sI_2 + \mu_1\tilde{M}_{q_1-1}(s))^{-1}. \end{aligned} \tag{38}$$

The matrices $\hat{Q}_{1,i}, 0 \le i \le 2$ are of the form

$$\hat{Q}_{1,0} = \begin{pmatrix} -\lambda & 0 \\ \mu_2 & -(\alpha_2+\lambda+\mu_2) \end{pmatrix}, \hat{Q}_{1,1} = \begin{pmatrix} -(\alpha_1+\lambda+\mu_1) & 0 \\ \mu_2 & -(\alpha+\lambda+\mu) \end{pmatrix}, \hat{Q}_{1,2} = \hat{Q}_{1,1} + \lambda\mathbf{e}_0(2) \otimes \mathbf{e}_1'(2).$$

The LST $\tilde{r}_{(q_1,1)}(s)$ can be calculated by the formula

$$\tilde{r}_{(q_1,1)}(s) = \frac{z(s)((1-z(s))\mu\mathbf{e}_1'(2)\tilde{L}_{q_1}(s) + \alpha)}{(1-z(s))(\lambda - z(s)\mu\mathbf{e}_1'(2)\tilde{M}_{q_1}(s))}, \tag{39}$$

and the function $z(s)$ is defined as

$$z(s) = \frac{s+\alpha+\lambda+\mu}{2\mu} - \sqrt{\left(\frac{s+\alpha+\lambda+\mu}{2\mu}\right)^2 - \frac{\lambda}{\mu}}. \tag{40}$$

Proof. Similarly we define the LSTs $\tilde{r}_x(s)$ for the PDF of first passage time T_x to the absorbing states of the process $\{\hat{X}_2(t)\}_{t\ge0}$ given the initial state is $x \in \hat{E}_2$. Consider now the states x with a queue length $q \ge q_1$. We again employ first-step analysis to obtain the system

$$-(s+\alpha+\lambda+\mu)\tilde{r}_{(q,1,1)}(s) + \lambda\tilde{r}_{(q+1,1,1)}(s) + \mu\tilde{r}_{(q-1,1,1)}(s) + \alpha = 0. \tag{41}$$

Now we rewrite system (41) in terms of the partial generating function $\tilde{P}_{1,1}(s,z)$ defined in the same way as in (29). This yields the following equation:

$$\tilde{P}_{1,1}(s,z) = \frac{(1-z)(z\mu\tilde{r}_{(q_1-1,1,1)}(s) - \lambda\tilde{r}_{(q_1,1,1)}(s)) + z\alpha}{(1-z)(-\mu z^2 + (s+\alpha+\lambda+\mu)z - \lambda)}. \tag{42}$$

The denominator in (42) is equal to zero if $z = 1$ or if $F(s,z) = -\mu z^2 + (s+\alpha+\lambda+\mu)z - \lambda = 0$. Note that $F(s,0) = -\lambda < 0$ and $F(s,1) = s+\alpha \ge 0$. Hence this for any $s > 0$ there is a minimal root $z = z(s) \in [0,1]$ for the equation $F(s,z) = 0$ which can be calculated by (40). This implies that the numerator of $\tilde{P}_{1,1}(s,z)$ at point $z(s)$ must also be zero and

$$(1-z(s))(z(s)\mu\tilde{r}_{(q_1-1,1,1)}(s) - \lambda\tilde{r}_{(q_1,1,1)}(s)) + z(s)\alpha = 0. \tag{43}$$

To obtain the second equation for the boundary transforms $\tilde{r}_{(q_1-1,1,1)}(s)$ and $\tilde{r}_{(q_1,1,1)}(s)$, we use the same procedure as in the previous proof. To this end, define the column vectors of the LSTs $\tilde{r}_x(s)$ for the states with a queue length $q \le q_1 - 1$:

$$\tilde{\mathbf{r}}_{0,0}(s) = (\tilde{r}_{(0,0,0)}(s), \tilde{r}_{(0,0,1)}(s))', \tilde{\mathbf{r}}_{q,1}(s) = (\tilde{r}_{(q,1,0)}(s), \tilde{r}_{(q,1,1)}(s))', 1 \le q \le q_1 - 1.$$

Applying first-step analysis to (27) yields the following recurrent relations:

$$\begin{aligned} &\tilde{\mathbf{r}}_{0,0}(s) = \tilde{M}_0(s)\tilde{\mathbf{r}}_{0,1}(s) + \tilde{L}_0(s), \\ &\tilde{\mathbf{r}}_{q,1}(s) = \tilde{M}_{q+1}(s)\tilde{\mathbf{r}}_{q+1,1}(s) + \tilde{L}_{q+1}(s), 0 \le q \le q_1 - 2, \\ &\tilde{\mathbf{r}}_{q_1-1,1}(s) = \tilde{M}_{q_1}(s)\tilde{r}_{(q_1,1,1)}(s) + \tilde{L}_{q_1}(s), \end{aligned} \tag{44}$$

where $\tilde{M}_i(s)$ and $\tilde{L}_i(s), 0 \le i \le q_1$, can be calculated by (38). Combining the last equation in (44),

$$\tilde{r}_{(q_1-1,1,1)}(s) = \mathbf{e}_1'(2)\tilde{\mathbf{r}}_{q_1-1,1}(s) = \mathbf{e}_1'(2)\tilde{M}_{q_1}(s)\tilde{r}_{(q_1,1,1)}(s) + \mathbf{e}_1'(2)\tilde{L}_{q_1}(s),$$

with (43), we obtain an explicit expression for the boundary transform $\tilde{r}_{(q_1,1,1)}(s)$ of the form (39). Further recursive substitution in (44) yields the relation

$$\tilde{\mathbf{r}}_{0,0}(s) = \prod_{j=0}^{q_1}\tilde{M}_j(s)\tilde{r}_{(q_1,1,1)}(s) + \sum_{i=0}^{q_1}\prod_{j=0}^{i-1}\tilde{M}_j(s)\tilde{L}_i(s), \tag{45}$$

which, together with (35) and $\tilde{r}_{(0,0,0)}(s) = \mathbf{e}'_0(2)\tilde{\mathbf{r}}_{0,0}(s)$, completes the proof. □

Theorem 5. *The Laplace transform of $R_3(t)$ is given by*

$$\tilde{R}_3(s) = \frac{1}{s}\left[1 - \mathbf{e}'_0(3)\prod_{j=0}^{q_1}\tilde{M}_j(s)\tilde{\mathbf{r}}_{q_1,1}(s) - \mathbf{e}'_0(3)\sum_{i=1}^{q_1}\prod_{j=0}^{i-1}\tilde{M}_j(s)\tilde{L}_i(s)\right], \tag{46}$$

where $\tilde{M}_i(s), 0 \le i \le q_1$, and $\tilde{L}_i(s), 1 \le i \le q_1$, are evaluated recursively,

$$\begin{aligned}
&\tilde{M}_0(s) = \lambda\tilde{N}_0(s), \tilde{N}_0(s) = -(\hat{Q}_{1,0} - sI_3)^{-1}, \\
&\tilde{M}_1(s) = \lambda\tilde{N}_1(s), \tilde{L}_1(s) = \alpha_1\tilde{N}_1\mathbf{e}(3), \tilde{N}_1 = -(\hat{Q}_{1,1} - sI_3 + \mu_1\tilde{M}_0(s))^{-1}, \\
&\tilde{M}_q(s) = \lambda\tilde{N}_q(s), \tilde{L}_q(s) = \tilde{N}_q(s)(\mu_1\tilde{L}_{q-1}(s) + \alpha_1\mathbf{e}(3)), 2 \le q \le q_1 - 1, \\
&\tilde{N}_q(s) = -(\hat{Q}_{1,1} - sI_3 + \mu_1\tilde{M}_{q-1}(s))^{-1}, 2 \le q \le q_1 - 1, \\
&\tilde{M}_{q_1}(s) = \tilde{N}_{q_1}(s)\hat{Q}_{0,1}, \tilde{L}_{q_1}(s) = \tilde{N}_{q_1}(s)(\mu_1\tilde{L}_{q_1-1}(s) + \alpha_1\mathbf{e}(3)), \\
&\tilde{N}_{q_1}(s) = -(\hat{Q}_{1,2} - sI_3 + \mu_1\tilde{M}_{q_1-1}(s))^{-1}.
\end{aligned} \tag{47}$$

The matrices $\hat{Q}_{1,i}, 0 \le i \le 2$ and $\hat{Q}_{0,1}$ are of the form

$$\hat{Q}_{1,0} = \begin{pmatrix} -\lambda & 0 & 0 \\ \mu_2 & -(\alpha_2+\lambda+\mu_2) & \alpha_2 \\ 0 & \beta_2 & -(\beta_2+\lambda) \end{pmatrix}, \hat{Q}_{1,1} = \begin{pmatrix} -(\alpha_1+\lambda+\mu_1) & 0 & 0 \\ \mu_2 & -(\alpha+\lambda+\mu) & \alpha_2 \\ 0 & \beta_2 & -(\alpha_1+\beta_2+\lambda+\mu_1) \end{pmatrix},$$

$$\hat{Q}_{1,2} = \hat{Q}_{1,1} + \lambda\mathbf{e}_0(3)\otimes\mathbf{e}'_1(3), \hat{Q}_{0,1} = \lambda\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

The column vector $\tilde{\mathbf{r}}_{q_1,1}(s) = (\tilde{r}_{(q_1,1,1)}(s), \tilde{r}_{(q_1,1,2)}(s))'$ is a solution of the system

$$\tilde{\mathbf{r}}_{q_1-1,1}(s) = \tilde{M}_{q_1}(s)\tilde{\mathbf{r}}_{q_1,1}(s) + \tilde{L}_{q_1}(s), \tag{48}$$

$$X_3(z)[X_4(z) + Y_2(s,z)] + Y_1(s,z)[X_2(s,z) + X_4(z)]\Big|_{z=z(s)} = 0, \tag{49}$$

$$X_1(s,z)[X_4(z) + Y_2(s,z)] + Y_3(z)[X_2(s,z) + X_4(z)]\Big|_{z=z(s)} = 0, \tag{50}$$

where

$$\begin{aligned}
&X_1(s,z) = (1-z)[(s+\alpha+\lambda+\mu)z - \lambda - \mu z^2], \\
&Y_1(s,z) = (1-z)[(s+\alpha_1+\beta_2+\lambda+\mu_1)z - \lambda - \mu_1 z^2], \\
&X_2(s,z) = \mu z(1-z)\tilde{r}_{(q_1-1,1,1)}(s) - \lambda(1-z)\tilde{r}_{(q_1,1,1)}(s), \\
&Y_2(s,z) = \mu_1 z(1-z)\tilde{r}_{(q_1-1,1,2)}(s) - \lambda(1-z)\tilde{r}_{(q_1,1,2)}(s), \\
&X_3(z) = \alpha_2 z(1-z), X_4(z) = \alpha_1 z, Y_3(z) = \beta_2 z(1-z),
\end{aligned} \tag{51}$$

and the function $z(s)$ is a minimal solution in the interval [0,1] of the equation

$$X_1(s,z)Y_1(s,z) - X_3(z)Y_3(z) = 0. \tag{52}$$

Proof. Let us denote by T_x the first passage time to absorbing states of the process $\{\hat{X}_3(t)\}_{t\ge0}$ given that the initial state is $x \in \hat{E}_3$ by $\tilde{r}_x(s)$ the LST of the corresponding PDF function. For the states x with a queue length above threshold level $q \ge q_1$ using (27) we obtain

$$\begin{aligned}
&-(s+\alpha+\lambda+\mu)\tilde{r}_{(q,1,1)}(s) + \alpha_1 + \alpha_2\tilde{r}_{(q,1,2)}(s) + \lambda\tilde{r}_{(q+1,1,1)}(s) + \mu\tilde{r}_{(q-1,1,1)}(s) = 0, \\
&-(s+\alpha_1+\beta_2+\lambda+\mu_1)\tilde{r}_{(q,1,2)}(s) = \alpha_1 + \beta_2\tilde{r}_{(q,1,1)}(s) + \lambda\tilde{r}_{(q+1,1,2)}(s) + \mu_1\tilde{r}_{(q-1,1,2)}(s) = 0.
\end{aligned} \tag{53}$$

In terms of the double transforms $\tilde{P}_{1,1}(s,z)$ and $\tilde{P}_{1,2}(s,z)$ defined in (29) the last system (53) can be rewritten in the form

$$\begin{aligned}
\tilde{P}_{1,1}(s,z) &= \frac{X_3(z)[X_4(z)+Y_2(s,z)] + Y_1(s,z)[X_2(s,z)+X_4(z)]}{X_1(s,z)Y_1(s,z) - X_3(z)Y_3(z)}, \\
\tilde{P}_{1,2}(s,z) &= \frac{X_1(s,z)[X_4(z)+Y_2(s,z)] + Y_3(z)[X_2(s,z)+X_4(z)]}{X_1(s,z)Y_1(s,z) - X_3(z)Y_3(z)},
\end{aligned} \tag{54}$$

where $X_i(s, z)$, $Y_i(s, z)$, $i \in \{1, 2\}$, $X_3(z)$, $Y_3(z)$ and $X_4(z)$ are defined in (51). For the denominator $F(s,z) = X_1(s,z)Y_1(s,z) - X_3(z)Y_3(z)$ the equation $F(s,z) = 0$ has for any $s > 0$ at least one root $z(s)$ in the interval $[0, \min\{z_1(s), z_2(s)\}]$, since $F(s,0) = \lambda^2 > 0$ and $F(s, z_i(s)) =$

$-\alpha_2\beta_2 z_i^2(s) \le 0$, where $z_1(s) \in [0, 1]$ and $z_2(s) \in [0, 1]$ are the roots of the polynomial equations $X_1(s, z) = 0$ and $Y_1(s, z) = 0$. Hence, at point $z = z(s)$ the numerators of $\tilde{P}_{1,1}(s, z)$ and $\tilde{P}_{1,2}(s, z)$ must also be equal to zero, and thus we obtain the two Eqs. (49) and (50) with four unknown boundary transforms $\tilde{r}_{(q,1,1)}$ and $\tilde{r}_{(q,1,2)}(s)$ for $q \in \{q_1 - 1, q_1\}$. To arrive at another two equations for these boundary transforms consider the states with a queue length below threshold level q_1, that is, for $q \le q_1$. Define the following column vectors of LSTs $\tilde{r}_x(s)$:

$$\tilde{\mathbf{r}}_{0,0}(s) = (\tilde{r}_{(0,0,0)}(s), \tilde{r}_{(0,0,1)}(s), \tilde{r}_{(0,0,2)}(s))', \tag{55}$$
$$\tilde{\mathbf{r}}_{q,1}(s) = \begin{cases} (\tilde{r}_{(q,1,0)}(s), \tilde{r}_{(q,1,1)}(s), \tilde{r}_{(q,1,2)}(s))', & 0 \le q \le q_1 - 1, \\ (\tilde{r}_{(q,1,1)}(s), \tilde{r}_{(q,1,2)}(s))', & q = q_1. \end{cases}$$

As before, these vectors can be represented in recursive form by means of the forward-elimination-backward-substitution method:

$$\tilde{\mathbf{r}}_{0,0} = \tilde{M}_0(s)\tilde{\mathbf{r}}_{0,1}(s), \tag{56}$$
$$\tilde{\mathbf{r}}_{q,1} = \tilde{M}_{q+1}(s)\tilde{\mathbf{r}}_{q+1,1}(s) + \tilde{L}_{q+1}(s), 0 \le q \le q_1 - 1,$$

where $\tilde{M}_i(s), 0 \le i \le q_1$, and $\tilde{L}_i(s), 1 \le i \le q_1$, can be calculated by (47). From (56) it follows that

$$\tilde{\mathbf{r}}_{q_1-1,1}(s) = \tilde{M}_{q_1}(s)\tilde{\mathbf{r}}_{q_1,1}(s) + \tilde{L}_{q_1}(s).$$

Thus, we obtain the two missing equations for the boundary transforms. Finally, the main result (46) of the present statement follows from the same arguments as used in previous proofs. □

Theorem 6. *The Laplace transform of $R_4(t)$ is given by*

$$\tilde{R}_4(s) = \frac{1}{s}\left[1 - \mathbf{e}_0'(2)\prod_{j=0}^{q_1}\tilde{M}_j(s)\tilde{\mathbf{r}}_{q_1,1}(s) - \mathbf{e}_0'(2)\sum_{i=0}^{q_1}\prod_{j=0}^{i-1}\tilde{M}_j(s)\tilde{L}_i(s)\right], \tag{57}$$

where $\tilde{M}_i(s), 0 \le i \le q_1$, and $\tilde{L}_i(s), 0 \le i \le q_1$, are evaluated recursively,

$$\tilde{M}_0(s) = \tilde{N}_0(s)\hat{Q}_{0,1}, \tilde{L}_0(s) = \alpha_2\tilde{N}_0(s)\mathbf{e}_1(2), \tilde{N}_0(s) = -(\hat{Q}_{1,0} - sI_2)^{-1}, \tag{58}$$
$$\tilde{M}_1(s) = \lambda\tilde{N}_1(s), \tilde{L}_1(s) = \tilde{N}_1(\hat{Q}_{2,1}\tilde{L}_0(s) + \alpha_2(\mathbf{e}_2(4) + \mathbf{e}_3(4))), \tilde{N}_1 = -(\hat{Q}_{1,1} - sI_4 + \hat{Q}_{2,1}\tilde{M}_0(s))^{-1},$$
$$\tilde{M}_q(s) = \lambda\tilde{N}_q(s), \tilde{L}_q(s) = \tilde{N}_q(s)(\hat{Q}_{2,2}\tilde{L}_{q-1}(s) + \alpha_2(\mathbf{e}_2(4) + \mathbf{e}_3(4))), 2 \le q \le q_2 - 1,$$
$$\tilde{N}_q(s) = -(\hat{Q}_{1,1} - sI_4 + \hat{Q}_{2,2}\tilde{M}_{q-1}(s))^{-1}, 2 \le q \le q_2 - 1,$$
$$\tilde{M}_{q_2}(s) = \tilde{N}_{q_2}(s)\hat{Q}_{0,2}, \tilde{L}_{q_2}(s) = \tilde{N}_{q_2}(s)(\hat{Q}_{2,2}\tilde{L}_{q_2-1}(s) + \alpha_2(\mathbf{e}_2(4) + \mathbf{e}_3(4))),$$
$$\tilde{N}_{q_2}(s) = -(\hat{Q}_{1,2} - sI_4 + \hat{Q}_{2,2}\tilde{M}_{q_2-1}(s))^{-1},$$
$$\tilde{M}_{q_2+1}(s) = \lambda\tilde{N}_{q_2+1}(s), \tilde{L}_{q_2+1}(s) = \tilde{N}_{q_2+1}(s)(\hat{Q}_{2,3}\tilde{L}_{q_2}(s) + \alpha_2(\mathbf{e}_1(3) + \mathbf{e}_2(3))),$$
$$\tilde{N}_{q_2+1}(s) = -(\hat{Q}_{1,3} - sI_3 + \hat{Q}_{2,3}\tilde{M}_{q_2}(s))^{-1},$$
$$\tilde{M}_q(s) = \lambda\tilde{N}_q(s), \tilde{L}_q(s) = \tilde{N}_q(s)(\hat{Q}_{2,4}\tilde{L}_{q-1}(s) + \alpha_2(\mathbf{e}_1(3) + \mathbf{e}_2(3))), q_2 + 2 \le q \le q_1 - 1,$$
$$\tilde{N}_q(s) = -(\hat{Q}_{1,3} - sI_3 + \hat{Q}_{2,4}\tilde{M}_{q-1}(s))^{-1}, q_2 + 2 \le q \le q_1 - 1,$$
$$\tilde{M}_{q_1}(s) = \tilde{N}_{q_1}(s)\hat{Q}_{0,3}, \tilde{L}_{q_1}(s) = \tilde{N}_{q_1}(s)(\hat{Q}_{2,4}\tilde{L}_{q_1-1}(s) + \alpha_2(\mathbf{e}_1(3) + \mathbf{e}_2(3))),$$
$$\tilde{N}_{q_1}(s) = -(\hat{Q}_{1,4} - sI_3 + \hat{Q}_{2,4}\tilde{M}_{q_1-1}(s))^{-1}.$$

The matrices $\hat{Q}_{1,i}, 0 \le i \le 4, \hat{Q}_{0,i}, 1 \le i \le 3$, and $\hat{Q}_{2,i}, 1 \le i \le 4$, are of the form

$$\hat{Q}_{1,0} = \begin{pmatrix} -\lambda & 0 \\ \mu_2 & -(\lambda + \alpha_2 + \mu_2) \end{pmatrix}, \hat{Q}_{1,1} = \begin{pmatrix} -(\alpha_1 + \lambda + \mu_1) & \alpha_1 & 0 & 0 \\ \beta_1 & -(\beta_1 + \lambda) & 0 & 0 \\ \mu_2 & 0 & -(\alpha + \lambda + \mu) & \alpha_1 \\ 0 & \mu_2 & \beta_1 & -(\alpha_2 + \beta_1 + \lambda + \mu_2) \end{pmatrix},$$

$$\hat{Q}_{1,2} = \hat{Q}_{1,1} + \lambda\mathbf{e}_1(4) \otimes \mathbf{e}_3'(4), \hat{Q}_{1,3} = \begin{pmatrix} -(\alpha_1 + \lambda + \mu_1) & 0 & 0 \\ \mu_2 & -(\alpha + \lambda + \mu) & \alpha_1 \\ 0 & \beta_1 & -(\alpha_2 + \beta_1 + \lambda + \mu_2) \end{pmatrix},$$

$$\hat{Q}_{1,4} = \hat{Q}_{1,3} + \lambda\mathbf{e}_0(3) \otimes \mathbf{e}_1'(3), \hat{Q}_{0,1} = \lambda\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \hat{Q}_{0,2} = \lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \hat{Q}_{0,3} = \lambda\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\hat{Q}_{2,1} = \begin{pmatrix} \mu_1 & 0 \\ 0 & 0 \\ 0 & \mu_1 \\ 0 & 0 \end{pmatrix}, \hat{Q}_{2,2} = \begin{pmatrix} \mu_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \hat{Q}_{2,3} = \begin{pmatrix} \mu_1 & 0 & 0 & \alpha_1 \\ 0 & 0 & \mu_1 & 0 \\ 0 & 0 & 0 & \mu_2 \end{pmatrix}, \hat{Q}_{2,4} = \begin{pmatrix} \mu_1 & 0 & \alpha_1 \\ 0 & \mu_1 & 0 \\ 0 & 0 & \mu_2 \end{pmatrix}.$$

The column vector $\tilde{\mathbf{r}}_{q_1,1}(s) = (\tilde{r}_{(q_1,1,1)}(s), \tilde{r}_{(q_1,2,1)}(s))'$ *is a solution of the system*

$$\tilde{\mathbf{r}}_{q_1-1,1}(s) = \tilde{M}_{q_1}(s)\tilde{\mathbf{r}}_{q_1,1}(s) + \tilde{L}_{q_1}(s), \tag{59}$$

$$X_3(z)[X_4(z)+Y_2(s,z)] + Y_1(s,z)[X_2(s,z)+X_4(z)]\Big|_{z=z(s)} = 0, \tag{60}$$

$$X_1(s,z)[X_4(z)+Y_2(s,z)] + Y_3(z)[X_2(s,z)+X_4(z)]\Big|_{z=z(s)} = 0, \tag{61}$$

where

$$\begin{aligned} X_1(s,z) &= (1-z)[(s+\alpha+\lambda+\mu)z - \lambda - \mu z^2], \\ Y_1(s,z) &= (1-z)[(s+\alpha_2+\beta_1+\lambda+\mu_2)z - \lambda - \mu_2 z^2], \\ X_2(s,z) &= \mu z(1-z)\tilde{r}_{(q_1-1,1,1)}(s) - \lambda(1-z)\tilde{r}_{(q_1,1,1)}(s), \\ Y_2(s,z) &= \mu_2 z(1-z)\tilde{r}_{(q_1-1,2,1)}(s) - \lambda(1-z)\tilde{r}_{(q_1,2,1)}(s), \\ X_3(z) &= \alpha_1 z(1-z), X_4(z) = \alpha_2 z, Y_3(z) = \beta_1 z(1-z), \end{aligned} \tag{62}$$

and the function z(s) is a minimal solution in the interval [0,1] of the equation

$$X_1(s,z)Y_1(s,z) - X_3(z)Y_3(z) = 0. \tag{63}$$

Proof. Let us denote by T_x the first passage time to absorbing states of the process $\{\hat{X}_4(t)\}_{t\ge 0}$ given that the initial state is $x \in \hat{E}_4$ and by $\tilde{r}_x(s)$ the LST of the corresponding PDF function. As before, the transforms $\tilde{r}_x(s)$ for the states with $q \ge q_1$ satisfy the system,

$$\begin{aligned} &-(s+\alpha+\lambda+\mu)\tilde{r}_{(q,1,1)}(s) + \alpha_1\tilde{r}_{(q,2,1)} + \alpha_2 + \lambda\tilde{r}_{(q+1,1,1)}(s) \\ &\quad + \mu\tilde{r}_{(q-1,1,1)}(s) = 0, \\ &-(s+\alpha_2+\beta_1+\lambda+\mu_2)\tilde{r}_{(q,2,1)}(s) = \alpha_2 + \beta_1\tilde{r}_{(q,1,1)}(s) \\ &\quad + \lambda\tilde{r}_{(q+1,2,1)}(s) + \mu_2\tilde{r}_{(q-1,2,1)}(s) = 0. \end{aligned}$$

In terms of the double transforms $\tilde{P}_{1,1}(s,z)$ and $\tilde{P}_{2,1}(s,z)$ defined in (29), the last system (64) can be rewritten in the form

$$\begin{aligned} \tilde{P}_{1,1}(s,z) &= \frac{X_3(z)[X_4(z)+Y_2(s,z)] + Y_1(s,z)[X_2(s,z)+X_4(z)]}{X_1(s,z)Y_1(s,z) - X_3(z)Y_3(z)}, \\ \tilde{P}_{2,1}(s,z) &= \frac{X_1(s,z)[X_4(z)+Y_2(s,z)] + Y_3(z)[X_2(s,z)+X_4(z)]}{X_1(s,z)Y_1(s,z) - X_3(z)Y_3(z)}, \end{aligned} \tag{65}$$

where $X_i(s, z)$, $Y_i(s, z)$, $i \in \{1, 2\}$, $X_3(z)$, $Y_3(z)$ and $X_4(z)$ are defined in (62). For the same reasons as in the previous proof, it can be shown, that for any s there is a value $z(s) \in [0, 1]$ where the denominator $F(s,z) = X_1(s,z)Y_1(s,z) - X_3(z)Y_3(z)$ is equal to zero. Hence, we obtain two relations (60) and (61) for four unknown boundary transforms $\tilde{r}_{(q,1,1)}$ and $\tilde{r}_{(q,2,1)}(s)$ for $q \in \{q_1-1, q_1\}$. To arrive at two more equations for the boundary transforms, the following column vectors of LSTs $\tilde{r}_x(s)$ are introduced:

$$\tilde{\mathbf{r}}_{0,0}(s) = (\tilde{r}_{(0,0,0)}(s), \tilde{r}_{(0,0,1)}(s))', \tag{66}$$

$$\tilde{\mathbf{r}}_{q,1}(s) = \begin{cases} (\tilde{r}_{(q,1,0)}(s), \tilde{r}_{(q,2,0)}(s), \\ \quad \tilde{r}_{(q,1,1)}(s), \tilde{r}_{(q,2,1)}(s))', & 0 \le q \le q_2 - 1, \\ (\tilde{r}_{(q,1,0)}(s), \tilde{r}_{(q,1,1)}(s), \tilde{r}_{(q,2,1)}(s))', & q_2 \le q \le q_1 - 1, \\ (\tilde{r}_{(q,1,1)}(s), \tilde{r}_{(q,2,1)}(s))', & q = q_1. \end{cases} \tag{66}$$

As previously, these vectors can be represented in the following recursive form:

$$\begin{aligned} \tilde{\mathbf{r}}_{0,0} &= \tilde{M}_0(s)\tilde{\mathbf{r}}_{0,1}(s) + \tilde{L}_0(s), \\ \tilde{\mathbf{r}}_{q,1} &= \tilde{M}_{q+1}(s)\tilde{\mathbf{r}}_{q+1,1}(s) + \tilde{L}_{q+1}(s), 0 \le q \le q_1 - 1, \end{aligned} \tag{67}$$

where $\tilde{M}_i(s)$ and $\tilde{L}_i(s), 0 \le i \le q_1$, are given by (58). From (67) it follows that

$$\tilde{\mathbf{r}}_{q_1-1,1}(s) = \tilde{M}_{q_1}(s)\tilde{\mathbf{r}}_{q_1,1}(s) + \tilde{L}_{q_1}(s).$$

Thus, we obtain the two missing equations for the boundary transforms. Finally, the main result (57) of the present statement follows from the same arguments as used before. □

Remark 2. The mean time to failure $\mathbb{E}[T_n]$ can be evaluated by $\mathbb{E}[T_n] = \lim_{s\to 0}\tilde{R}_n(s), 1 \le n \le 4$. The algorithms for evaluating the LST $\tilde{r}_x(s)$ proposed above can be modified to calculate the corresponding moments $\bar{m}_x$ using the system

$$\bar{m}_x = \frac{1}{\lambda_x}\Big(1 + \sum_{y\ne x}\lambda_{xy}\bar{m}_y\Big), x \in \hat{E}_n.$$

For example, the mean time to first failure $\mathbb{E}[T_1]$ satisfies

$$\mathbb{E}[T_1] = \mathbf{e}_0'(3)\left[\prod_{j=0}^{q_1}\tilde{M}_j(0)\bar{\mathbf{m}}_{q_1,1} + \sum_{i=0}^{q_1}\prod_{j=0}^{i-1}\tilde{M}_j(0)\tilde{L}_i(0)\right], \tag{68}$$

where

$$\tilde{M}_0(0) = \tilde{N}_0(0)\hat{Q}_{0,1}, \tilde{L}_0(0) = \tilde{N}_0(0)\mathbf{e}(3), \tilde{N}_0(0) = -\hat{Q}_{1,0}^{-1}, \tag{69}$$

and the remaining matrices $\tilde{M}_i(0)$ and $\tilde{L}_i(0), 1 \le i \le q_1$, are defined by (20) at point $s = 0$, but the sum of vectors with factors α_j replaced by $\mathbf{e}$. The column vector $\bar{\mathbf{m}}_{q_1,1} = (\bar{m}_{(q_1,1,1)}, \bar{m}_{(q_1,2,1)}, \bar{m}_{(q_1,1,2)})'$ is a solution of the system

$$\begin{aligned} &\bar{\mathbf{m}}_{q_1-1,1} = \tilde{M}_{q_1}(0)\bar{\mathbf{m}}_{q_1,1} + \tilde{L}_{q_1}(0), \\ &Z_1(z)[X_3(z)(Y_2(z)+z) + Y_1(z)(X_2(z)+z)] \\ &\quad + X_4(z)Y_1(z)[Z_2+z]\Big|_{z=z'} = 0, \\ &Z_1(z)[X_1(z)(Y_2(z)+z) + Y_3(z)(X_2(z)+z)] \\ &\quad + X_4(z)Y_3(z)[Z_2+z] - X_4(z)Z_3(z)[Y_2+z]\Big|_{z=z'} = 0, \\ &X_1(z)Y_1(z)[Z_2+z] - X_3(z)Y_3(z)[Z_2(z)+z] \\ &\quad + Y_1(z)Z_3(z)[X_2(z)+z] + X_3(z)Z_3(z)[Y_2(z)+z]\Big|_{z=z'} = 0, \end{aligned} \tag{70}$$

where

$$\begin{aligned} X_1(z) &= (1-z)[(\alpha+\lambda+\mu)z - \lambda - \mu z^2], \\ Y_1(z) &= (1-z)[(\alpha_2+\beta_1+\lambda+\mu_2)z - \lambda - \mu_2 z^2], \\ Z_1(z) &= (1-z)[(\alpha_1+\beta_2+\lambda+\mu_1)z - \lambda - \mu_1 z^2], \\ X_2(z) &= \mu z(1-z)\bar{m}_{(q_1-1,1,1)} - \lambda(1-z)\bar{m}_{(q_1,1,1)}, \\ Y_2(z) &= \mu_2 z(1-z)\bar{m}_{(q_1-1,2,1)} - \lambda(1-z)\bar{m}_{(q_1,2,1)}, \\ Z_2(z) &= \mu_1 z(1-z)\bar{m}_{(q_1-1,1,2)} - \lambda(1-z)\bar{m}_{(q_1,1,2)}, \\ X_3(z) &= \alpha_1 z(1-z), X_4(z) = \alpha_2 z(1-z), \\ Y_3(z) &= \beta_1 z(1-z), Z_3(z) = \beta_2 z(1-z), \end{aligned} \tag{71}$$

and the function z' is a minimal solution in the interval [0,1] of the equation

$$X_1(z)Y_1(z)Z_1(z) - X_3(z)Y_3(z)Z_1(z) - X_4(z)Y_1(z)Z_3(z) = 0. \tag{72}$$

Similarly, we obtain expressions for the means $\mathbb{E}[T_n], 2 \le n \le 4$.

4. The number of failures (repairs) during a life time

In this section, we study the number of failures (repairs) during life times T_1, T_3 and T_4. This complements the reliability analysis and provides a type of a discrete counterpart of the length of life time. The methodology is similar to that employed in the previous section, and hence we omit some repetitive details.

Theorem 7. *The generating function* $\tilde{\psi}_{1,1}(z)$ *of the number of failures of server 1 during* T_1 *is calculated by*

$$\tilde{\psi}_{1,1}(z) = \mathbf{e}_0'(3)\left[\prod_{j=0}^{q_1}\tilde{M}_j(z)\tilde{\boldsymbol{\psi}}_{q_1,1}(z) + \sum_{i=1}^{q_1}\prod_{j=0}^{i-1}\tilde{M}_j(z)\tilde{L}_i(z)\right], \tag{73}$$

where $\tilde{M}_i(z), 0 \le i \le q_1$, and $\tilde{L}_i(z), 1 \le i \le q_1$, satisfy the same recursive relations (20) at point $s = 0$ with the failure rate α_1 replaced by $\alpha_1 z$. The matrices $\hat{Q}_{1,i}, 0 \le i \le 4$ and $\hat{Q}_{2,i}, i = 3, 4$, are substituted respectively with $\hat{Q}_{1,i}(z)$ and $\hat{Q}_{2,i}(z)$, where the rates α_1 (with the exception of the main diagonals in matrices $\hat{Q}_{1,i}$) are replaced by $\alpha_1 z$. The column vector $\tilde{\boldsymbol{\psi}}_{q_1,1}(z) = (\tilde{\psi}_{(q_1,1,1)}(z), \tilde{\psi}_{(q_1,2,1)}(z), \tilde{\psi}_{(q_1,1,2)}(z))'$ is a solution of the system

$$\tilde{\boldsymbol{\psi}}_{q_1-1,1}(z) = \tilde{M}_{q_1}(z)\tilde{\boldsymbol{\psi}}_{q_1,1}(z) + \tilde{L}_{q_1}(z), \tag{74}$$

$$X_3(z,z_1)X_4(z_1)Y_1(z_1) + X_3(z,z_1)Z_1(z_1)[X_4(z_1) + Y_2(z,z_1)] + Y_1(z_1)[X_2(z,z_1)Z_1(z_1) + X_4(z_1)Z_2(z,z_1)]\Big|_{z_1=z_1(z)} = 0, \tag{75}$$

$$X_3(z,z_1)X_4(z_1)Y_3(z_1) + X_1(z_1)Z_1(z_1)[X_4(z_1) + Y_2(z,z_1)] + Y_3(z_1)[X_2(z,z_1)Z_1(z_1) + X_4(z_1)Z_2(z,z_1)] - X_4(z_1)Z_3(z_1)[X_4(z_1) + Y_2(z,z_1)]\Big|_{z_1=z_1(z)} = 0, \tag{76}$$

$$X_2(z,z_1)Y_1(z_1)Z_3(z_1) + [X_3(z,z_1) + Z_2(z,z_1)][X_1(z_1)Y_1(z_1) - X_3(z,z_1)Y_3(z_1)] + X_3(z,z_1)Z_3(z_1)[X_4(z_1) + Y_2(z,z_1)]\Big|_{z_1=z_1(z)} = 0, \tag{77}$$

where

$$X_1(z_1) = (\alpha + \lambda + \mu)z_1 - \lambda - \mu z_1^2, \tag{78}$$
$$Y_1(z_1) = (1 - z_1)[(\alpha_2 + \beta_1 + \lambda + \mu_2)z_1 - \lambda - \mu_2 z_1^2],$$
$$Z_1(z_1) = (1 - z_1)[(\alpha_1 + \beta_2 + \lambda + \mu_1)z_1 - \lambda - \mu_1 z_1^2],$$
$$X_2(z,z_1) = \mu z_1 \tilde{\psi}_{(q_1-1,1,1)}(z) - \lambda \tilde{\psi}_{(q_1,1,1)}(z),$$
$$Y_2(z,z_1) = \mu_2 z_1 (1 - z_1)\tilde{\psi}_{(q_1-1,2,1)}(z) - \lambda(1 - z_1)\tilde{\psi}_{(q_1,2,1)}(z),$$
$$Z_2(z,z_1) = \mu_1 z_1 (1 - z_1)\tilde{\psi}_{(q_1-1,1,2)}(z) - \lambda(1 - z_1)\tilde{\psi}_{(q_1,1,2)}(z),$$
$$X_3(z,z_1) = \alpha_1 z z_1, X_4(z_1) = \alpha_2 z_1,$$
$$Y_3(z_1) = \beta_1 z_1(1 - z_1), Z_3(z_1) = \beta_2 z_1(1 - z_1),$$

and the function $z_1(z)$ is a minimal solution in the interval [0,1] of the equation

$$X_1(z_1)Y_1(z_1)Z_1(z_1) - X_3(z,z_1)Y_3(z_1)Z_1(z_1) - X_4(z_1)Y_1(z_1)Z_3(z_1) = 0. \tag{79}$$

Proof. First we introduce some notation:

Ψ_x – the number of failures of server 1 up to the end of the life time T_1 of the process $\{\hat{X}_1(t)\}$ given that the initial state is $x \in \hat{E}_1$;

$\psi_x(k) = \mathbb{P}[\Psi_x = k]$ – the probability density function (PDF) of Ψ_x;

$\tilde{\psi}_x(z) = \sum_{k=1}^{\infty} \psi_x(t)z^k, |z| \le 1$ – the probability generating function (PGF) of $\psi_x(t)$.

According to the law of the total probability, the density $\psi_x(t)$ satisfies the following system:

$$\psi_x(k) = \frac{\lambda_{xy'}}{\lambda_x}\psi_{y'}(k-1) + \sum_{y \ne x,y'} \frac{\lambda_{xy}}{\lambda_x}\psi_y(k), x \in \hat{E}_1.$$

The first term on the right-hand side stands for the transition to the state y', which we count (i.e., failures of server 1), while the second term includes other possible transitions. In terms of the PGF, the last system can be rewritten as follows:

$$\tilde{\psi}_x(z) = \frac{z\lambda_{xy'}}{\lambda_x}\tilde{\psi}_{y'}(z) + \sum_{y \ne x,y'} \frac{\lambda_{xy}}{\lambda_x}\tilde{\psi}_y(z), x \in \hat{E}_1. \tag{80}$$

Note that the system is the same as (27) for $s = 0$ but with α_1 replaced by $\alpha_1 z$. Employing the same steps as presented in the proof of Theorem 3, we obtain the corresponding result for the function $\tilde{\psi}_{1,1}(z)$. □

Similarly, we obtain the result for server 2. In this case, the system for the PGFs has the form (27) for $s = 0$, but the failure rates α_2 are replaced by $\alpha_2 z$. The corresponding result is summarized below.

Theorem 8. *The generating function $\tilde{\psi}_{1,2}(z)$ of the number of failures of server 2 during T_1 is calculated by*

$$\tilde{\psi}_{1,2}(z) = \mathbf{e}_0'(3)\left[\prod_{j=0}^{q_1}\tilde{M}_j(z)\tilde{\boldsymbol{\psi}}_{q_1,1}(z) + \sum_{i=1}^{q_1}\prod_{j=0}^{i-1}\tilde{M}_j(z)\tilde{L}_i(z)\right], \tag{81}$$

where $\tilde{M}_i(z), 0 \le i \le q_1$, and $\tilde{L}_i(z), 1 \le i \le q_1$, satisfy the same recursive relations (20) at point $s = 0$ with the failure rate α_2 replaced by $\alpha_2 z$. The matrices $\hat{Q}_{1,i}, 0 \le i \le 4$, are substituted by $\hat{Q}_{1,i}(z)$, where the rate α_2 (with the exception of the main diagonals) is replaced by $\alpha_2 z$. The column vector $\tilde{\boldsymbol{\psi}}_{q_1,1}(z) = (\tilde{\psi}_{(q_1,1,1)}(z), \tilde{\psi}_{(q_1,2,1)}(z), \tilde{\psi}_{(q_1,1,2)}(z))'$ is a solution of the system

$$\tilde{\boldsymbol{\psi}}_{q_1-1,1}(z) = \tilde{M}_{q_1}(z)\tilde{\boldsymbol{\psi}}_{q_1,1}(z) + \tilde{L}_{q_1}(z), \tag{82}$$

$$X_3(z_1)X_4(z,z_1)Y_1(z_1) + X_3(z_1)Z_1(z_1)[X_4(z,z_1) + Y_2(z,z_1)] + Y_1(z_1)[X_2(z,z_1)Z_1(z_1) + X_4(z,z_1)Z_2(z,z_1)]\Big|_{z_1=z_1(z)} = 0,$$

$$X_3(z_1)X_4(z,z_1)Y_3(z_1) + X_1(z_1)Z_1(z_1)[X_4(z_1) + Y_2(z,z_1)] + Y_3(z_1)[X_2(z,z_1)Z_1(z_1) + X_4(z,z_1)Z_2(z,z_1)] - X_4(z,z_1)Z_3(z_1)[X_4(z,z_1) + Y_2(z,z_1)]\Big|_{z_1=z_1(z)} = 0,$$

$$X_2(z,z_1)Y_1(z_1)Z_3(z_1) + [X_3(z_1) + Z_2(z,z_1)][X_1(z_1)Y_1(z_1) - X_3(z_1)Y_3(z_1)] + X_3(z_1)Z_3(z_1)[X_4(z,z_1) + Y_2(z,z_1)]\Big|_{z_1=z_1(z)} = 0,$$

where $X_1(z_1)$, $Y_1(z_1)$, $Z_1(z_1)$, $X_2(z, z_1)$, $Y_2(z, z_1)$, $Z_2(z, z_1)$, $Y_3(z_1)$ and $Z_3(z_1)$ are defined by (78), and $X_3(z_1) = \alpha_1 z_1, X_4(z,z_1) = \alpha_2 z z_1$. The function $z_1(z)$ is a minimal solution in the interval [0,1] of the equation

$$X_1(z_1)Y_1(z_1)Z_1(z_1) - X_3(z_1)Y_3(z_1)Z_1(z_1) - X_4(z,z_1)Y_1(z_1)Z_3(z_1) = 0. \tag{83}$$

Similarly, we obtain the corresponding PGFs for the number of failures of server 2 during the life time T_3 and of server 1 during the life time T_4. The results are summarized in the following two theorems:

Theorem 9. *The generating function $\tilde{\psi}_3(z)$ of the number of failures of server 2 during T_3 is calculated by*

$$\tilde{\psi}_3(z) = \mathbf{e}_0'(3)\left[\prod_{j=0}^{q_1}\tilde{M}_j(z)\tilde{\boldsymbol{\psi}}_{q_1,1}(z) + \sum_{i=1}^{q_1}\prod_{j=0}^{i-1}\tilde{M}_j(z)\tilde{L}_i(z)\right], \tag{84}$$

where $\tilde{M}_i(z), 0 \le i \le q_1$, and $\tilde{L}_i(z), 1 \le i \le q_1$, satisfy the same recursive relations (47) at point $s = 0$. The matrices $\hat{Q}_{1,i}, 0 \le i \le 4$, are substituted by $\hat{Q}_{1,i}(z)$, where the rate α_2 (with the exception of the main diagonals) is replaced by $\alpha_2 z$. The column vector $\tilde{\boldsymbol{\psi}}_{q_1,1}(z) = (\tilde{\psi}_{(q_1,1,1)}(z), \tilde{\psi}_{(q_1,1,2)}(z))'$ is a solution of the system

$$\tilde{\boldsymbol{\psi}}_{q_1-1,1}(z) = \tilde{M}_{q_1}(z)\tilde{\boldsymbol{\psi}}_{q_1,1}(z) + \tilde{L}_{q_1}(z), \tag{85}$$

$$X_3(z,z_1)[X_4(z_1)+Y_2(z,z_1)]+Y_1(z_1)[X_2(z,z_1)+X_4(z_1)]\Big|_{z_1=z_1(z)} = 0,$$

$$X_1(z_1)[X_4(z_1) + Y_2(z,z_1)] + Y_3(z_1)[X_2(z,z_1) + X_4(z_1)]\Big|_{z_1=z_1(z)} = 0,$$

where

$$X_1(z_1) = (1-z_1)[(\alpha+\lambda+\mu)z_1 - \lambda - \mu z_1^2], \tag{86}$$
$$Y_1(z_1) = (1-z_1)[(\alpha_1+\beta_2+\lambda+\mu_1)z_1 - \lambda - \mu_1 z_1^2],$$
$$X_2(z,z_1) = \mu z_1(1-z_1)\tilde{\psi}_{(q_1-1,1,1)}(z) - \lambda(1-z_1)\tilde{\psi}_{(q_1,1,1)}(z),$$
$$Y_2(z,z_1) = \mu_1 z_1(1-z_1)\tilde{\psi}_{(q_1-1,1,2)}(z) - \lambda(1-z_1)\tilde{\psi}_{(q_1,1,2)}(z),$$
$$X_3(z,z_1) = \alpha_2 z z_1(1-z_1), X_4(z_1) = \alpha_1 z_1, Y_3(z_1) = \beta_2 z_1(1-z_1).$$

The function $z_1(z)$ is a minimal solution in the interval [0,1] of the equation

$$X_1(z_1)Y_1(z_1) - X_3(z,z_1)Y_3(z_1) = 0. \tag{87}$$

Theorem 10. *The generating function $\tilde{\psi}_4(z)$ of the number of failures of server 1 during T_4 is calculated by*

$$\tilde{\psi}_4(z) = \mathbf{e}_0'(2)\left[\prod_{j=0}^{q_1}\tilde{M}_j(z)\tilde{\boldsymbol{\psi}}_{q_1,1}(z) + \sum_{i=0}^{q_1}\prod_{j=0}^{i-1}\tilde{M}_j(z)\tilde{L}_i(z)\right], \tag{88}$$

where $\tilde{M}_i(z)$ and $\tilde{L}_i(z), 0 \le i \le q_1$, satisfy the same recursive relations (58) at point $s = 0$. The matrices $\hat{Q}_{1,i}, 0 \le i \le 4$, and $\hat{Q}_{2,i}, i = 3, 4$, are substituted by $\hat{Q}_{1,i}(z)$ and $\hat{Q}_{2,i}(z)$, where the rate α_1 (with the exception of the main diagonals) is replaced by $\alpha_1 z$. The column vector $\tilde{\boldsymbol{\psi}}_{q_1,1}(z) = (\tilde{\psi}_{(q_1,1,1)}(z), \tilde{\psi}_{(q_1,2,1)}(z))'$ is a solution of the system

$$\tilde{\boldsymbol{\psi}}_{q_1-1,1}(z) = \tilde{M}_{q_1}(z)\tilde{\boldsymbol{\psi}}_{q_1,1}(z) + \tilde{L}_{q_1}(z), \tag{89}$$
$$X_3(z,z_1)[X_4(z_1)+Y_2(z,z_1)] + Y_1(z_1)[X_2(z,z_1)+X_4(z_1)]\Big|_{z_1=z_1(z)} = 0,$$
$$X_1(z_1)[X_4(z_1)+Y_2(z,z_1)] + Y_3(z_1)[X_2(z,z_1)+X_4(z_1)]\Big|_{z_1=z_1(z)} = 0,$$

where

$$X_1(z_1) = (1-z_1)[(\alpha+\lambda+\mu)z_1 - \lambda - \mu z_1^2], \tag{90}$$
$$Y_1(z_1) = (1-z_1)[(\alpha_2+\beta_1+\lambda+\mu_2)z_1 - \lambda - \mu_2 z_1^2],$$
$$X_2(z,z_1) = \mu z_1(1-z_1)\tilde{\psi}_{(q_1-1,1,1)}(z) - \lambda(1-z_1)\tilde{\psi}_{(q_1,1,1)}(z),$$
$$Y_2(z,z_1) = \mu_2 z_1(1-z_1)\tilde{\psi}_{(q_1-1,2,1)}(z) - \lambda(1-z_1)\tilde{\psi}_{(q_1,2,1)}(z),$$
$$X_3(z,z_1) = \alpha_1 z z_1(1-z_1), X_4(z_1) = \alpha_2 z_1, Y_3(z_1) = \beta_1 z_1(1-z_1),$$

and the function $z_1(z)$ is a minimal solution in the interval [0,1] of the equation

$$X_1(z_1)Y_1(z_1) - X_3(z,z_1)Y_3(z_1) = 0. \tag{91}$$

Remark 3. The mean number of failures $\mathbb{E}[\Psi_{1,1}], \mathbb{E}[\Psi_{1,2}], \mathbb{E}[\Psi_3]$ and $\mathbb{E}[\Psi_4]$ can be evaluated by $\mathbb{E}[\Psi_{1,1}] = \frac{d\tilde{\psi}_{1,1}(z)}{dz}\Big|_{z=1}$, $\mathbb{E}[\Psi_{1,2}] = \frac{d\tilde{\psi}_{1,2}(z)}{dz}\Big|_{z=1}$, $\mathbb{E}[\Psi_3] = \frac{d\tilde{\psi}_3(z)}{dz}\Big|_{z=1}$ and $\mathbb{E}[\Psi_4] = \frac{d\tilde{\psi}_4(z)}{dz}\Big|_{z=1}$. The algorithms for evaluating PGFs $\tilde{\psi}_x(z)$ proposed above can be modified to calculate the corresponding moments $\bar{\psi}_x$ using the system

$$\bar{\psi}_x = \frac{1}{\lambda_x}\left(\lambda_{xy'} + \sum_{y \ne x}\lambda_{xy}\bar{\psi}_y\right), x \in \hat{E}_n, 1 \le n \le 4.$$

5. Numerical results

In this section, we present some numerical examples to study the effect of system parameters on the proposed reliability measures. First, we fix the system parameters at values

$$\lambda = 1.7, \mu_1 = 2.4, \mu_2 = 0.4, \alpha_1 = 0.1, \alpha_2 = 0.2, \beta_1 = 0.3,$$
$$\beta_2 = 0.3, \rho = 0.83, q_1 = 9, q_2 = 6.$$

In all cases presented below the parametric values are chosen such that the ergodicity condition holds.

In Figs. 2a, 2b, 3a and 3b, the stationary availabilities A_i, $1 \le i \le 4$, are plotted against the arrival rate λ for failure rates α_1, α_2 and repair rates β_1, β_2. As expected, A_i decreases with increasing λ. The figures reflect dependences of availabilities A_i on failure and repair rates. The upper curves correspond to lower values of α_1 and α_2 and to higher values of $\beta_1$and β_2. We notice that descriptor A_3, which specifies the stationary availability of the first server, changes with varying α_1 and β_1, but it is insensitive to reliability characteristics α_2 and β_2 of the second server. This is caused by a threshold policy which prescribes to use the faster server whenever it is free and therefore the time to failure of this server is independent of the reliability attributes of the slower server.

In Figs. 4a, 4b, 5a and 5b, we plot the failure frequencies B_l for $\alpha_l = 0.1, 0.2, 0.3$ and $\beta_l = 0.2, 0.3, 0.4, l = 1, 2$. These characteristics increase monotonically with increasing λ. Further, we notice that $B_1 > B_2$, because the probability to be in state x with $d_1(x) = 1$ is higher than the probability for $d_2(x) = 1$, since server 2 is used according to the threshold control policy. We observe that function B_1 is insensitive to changes in α_2, β_1 and β_2, and function B_2 is almost insensitive to changes in β_2.

In Figs. 6a, 6b, 7a and 7b, we analyse the effect of the arrival rate λ on the reliability functions $R_1(t)$, $R_2(t)$, $R_3(t)$ and $R_4(t)$,

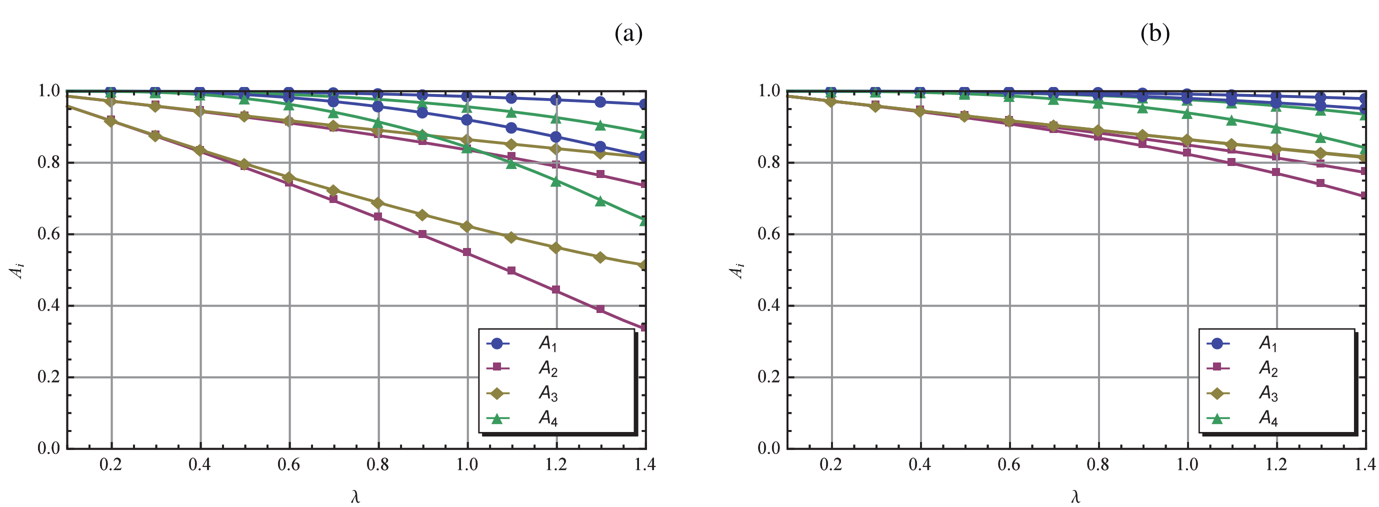

Fig. 2. The availability A_i, $1 \le i \le 4$, for $\alpha_1 = 0.1, 0.3$ (a) and $\alpha_2 = 0.1, 0.3$ (b).

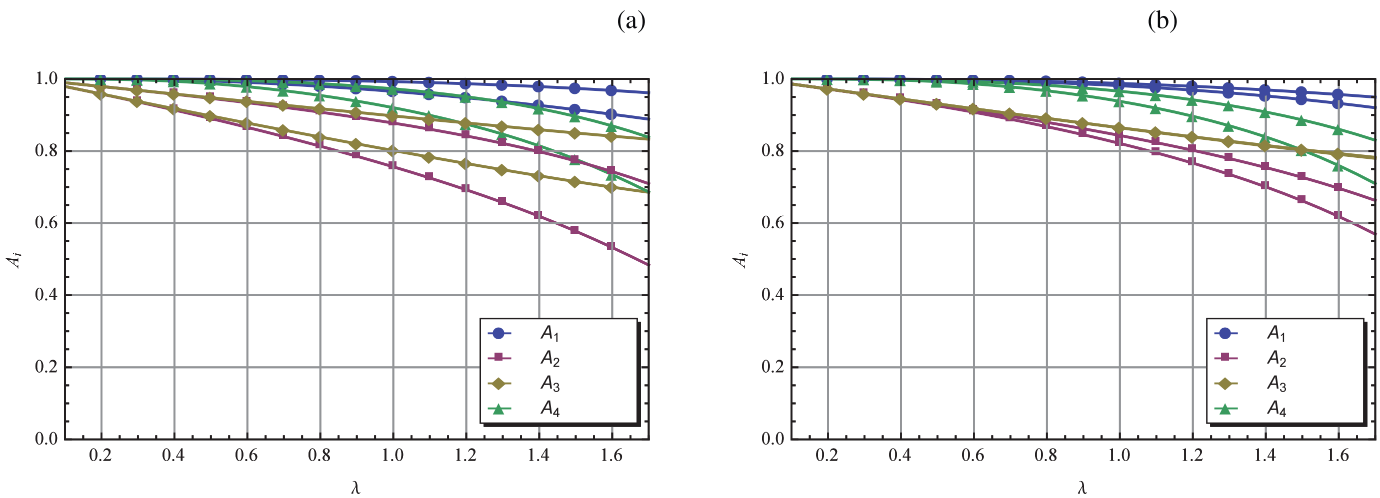

Fig. 3. The availability A_i, $1 \le i \le 4$, for $\beta_1 = 0.2, 0.4$ (a) and $\beta_2 = 0.2, 0.4$ (b).

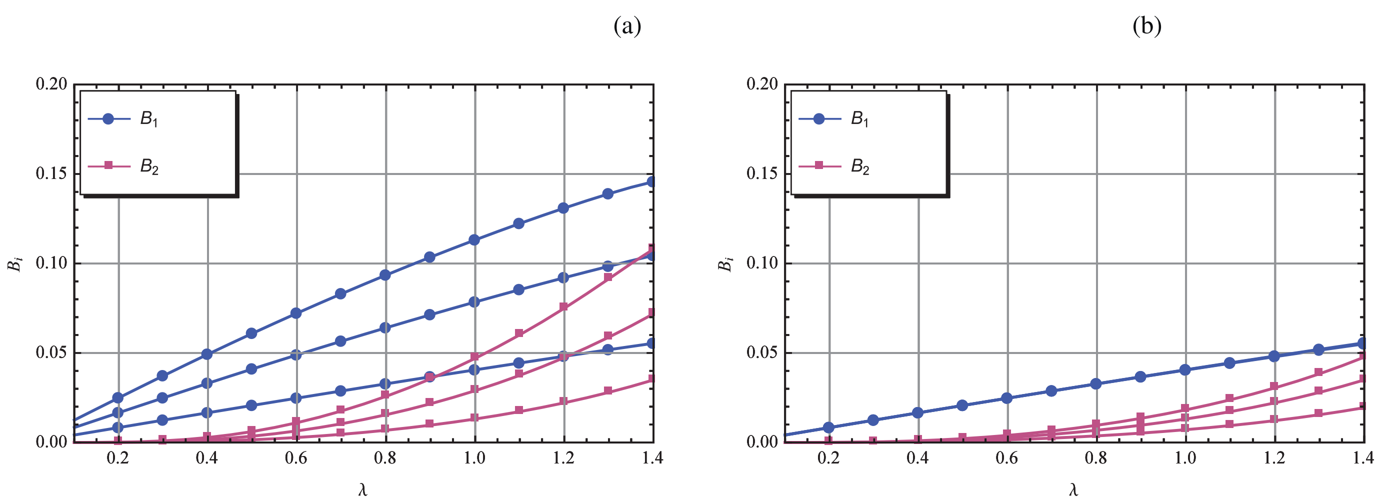

Fig. 4. The failure frequency $B_i, i = 1, 2,$ for $\alpha_1 = 0.1, 0.2, 0.3$ (a) and $\alpha_2 = 0.1, 0.2, 0.3$ (b) versus λ.

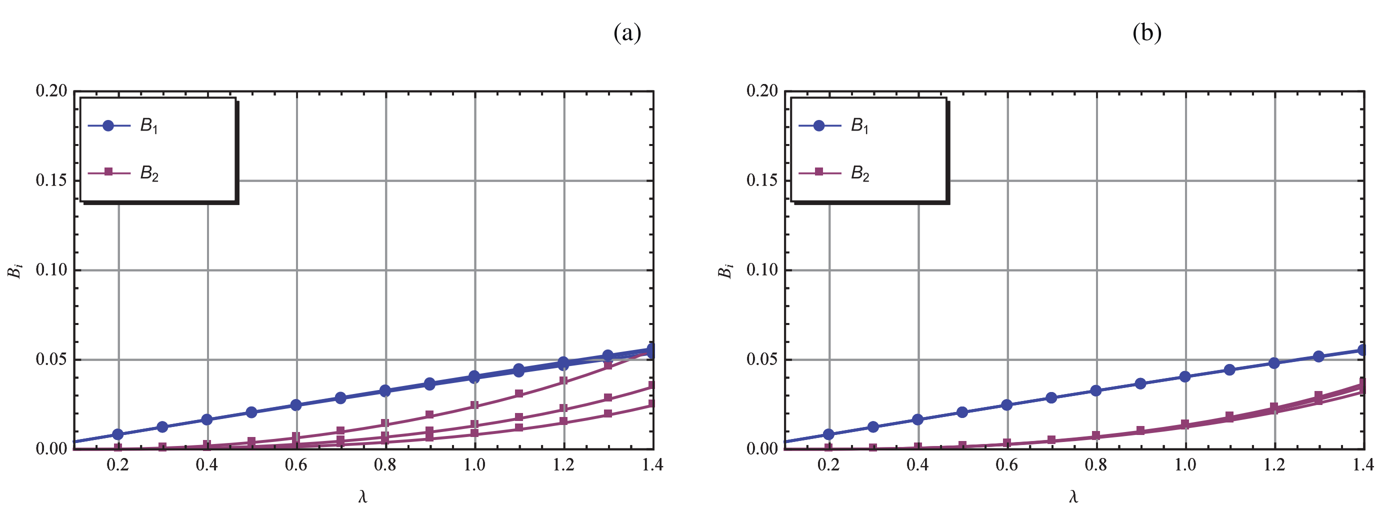

Fig. 5. The failure frequency $B_i, i = 1, 2,$ for $\beta_1 = 0.2, 0.3, 0.4$ (a) and $\beta_2 = 0.2, 0.3, 0.4$ (b) versus λ.

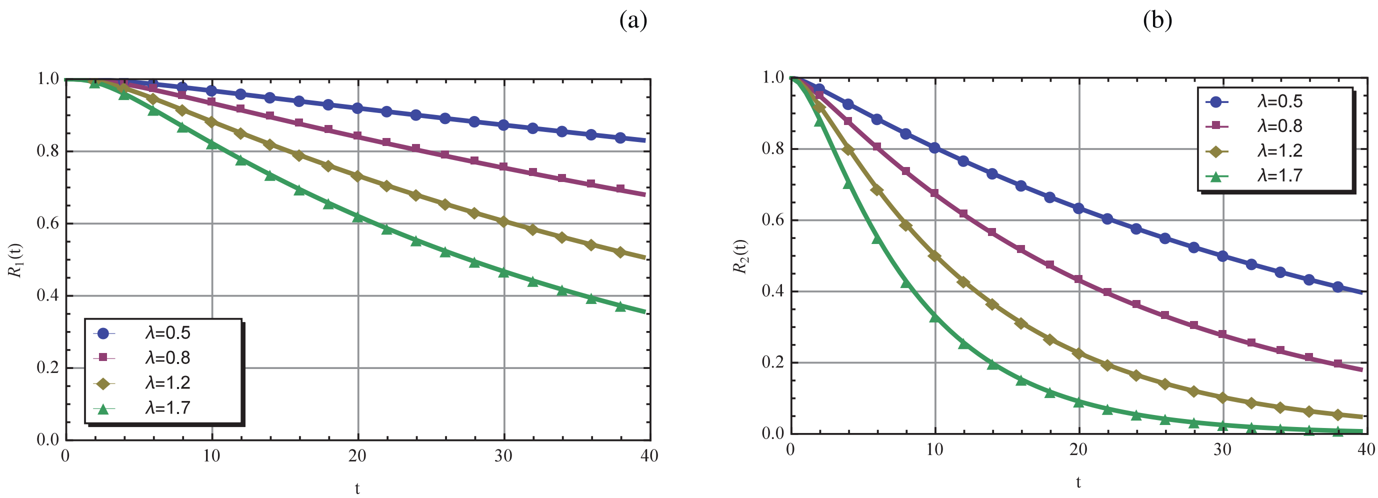

Fig. 6. The function $R_1(t)$ (a) and $R_2(t)$ (b).

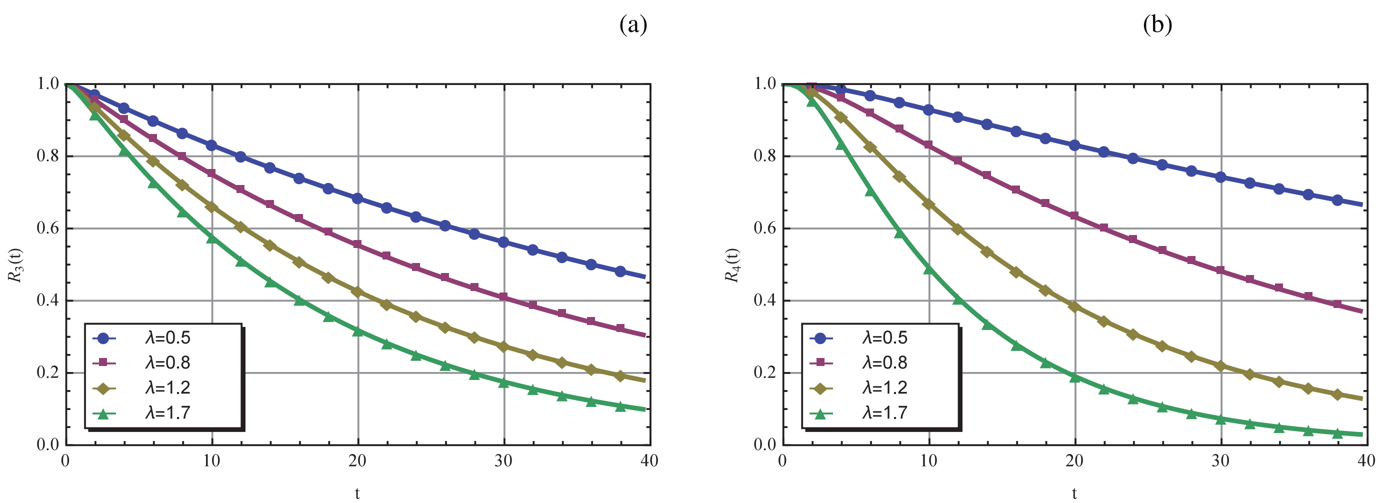

Fig. 7. The function $R_3(t)$ (a) and $R_4(t)$ (b).

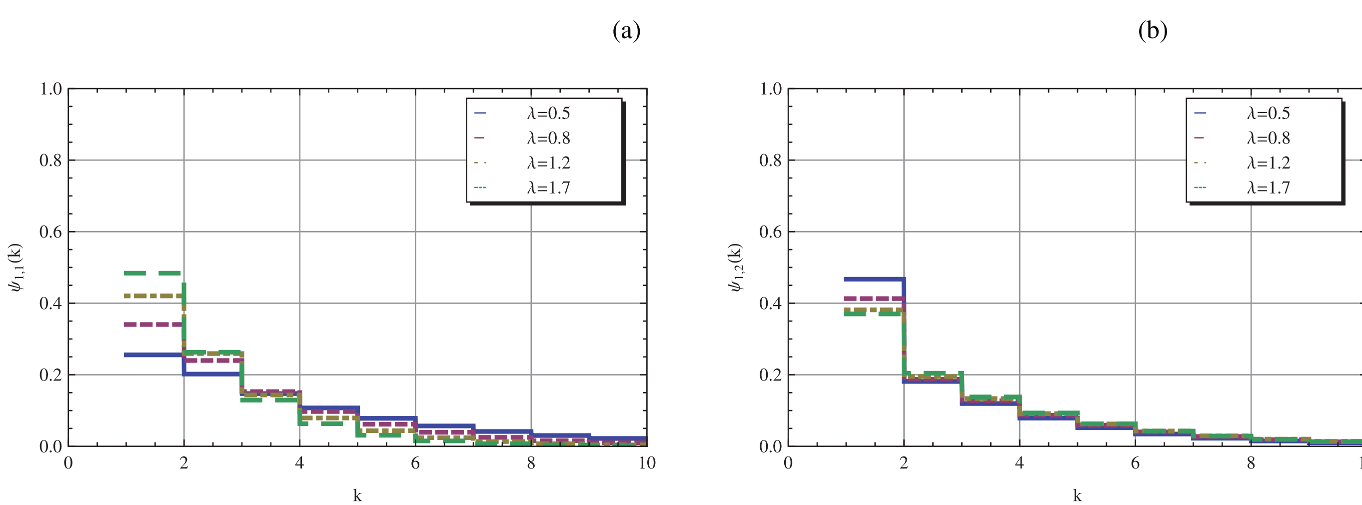

Fig. 8. The function $\psi_{1,1}(k)$ (a) and $\psi_{1,2}(k)$ (b).

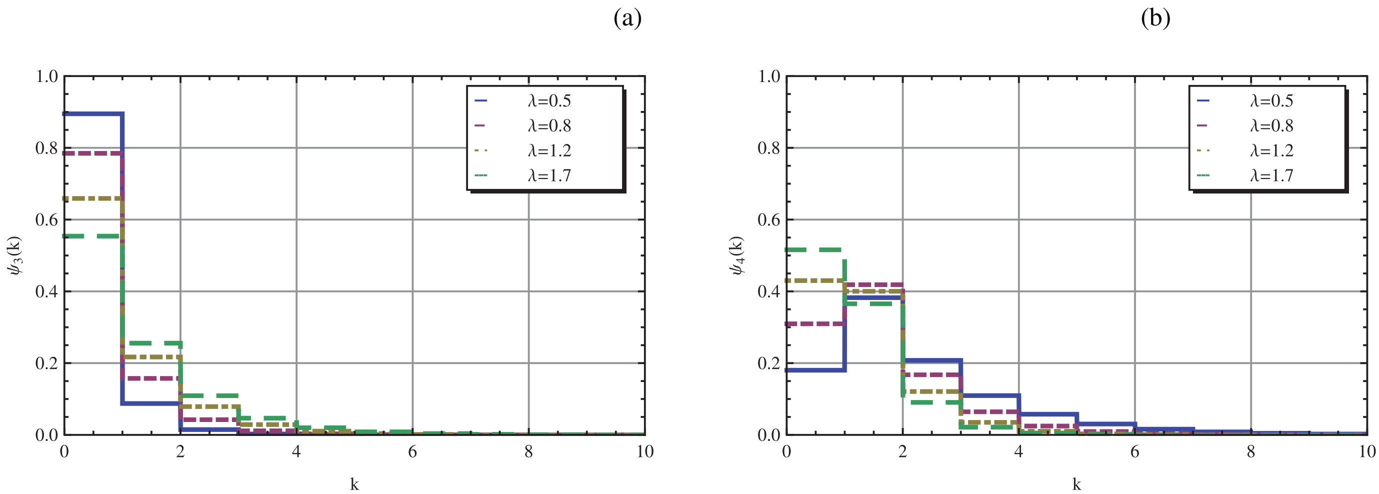

Fig. 9. The function $\psi_3(k)$ (a) and $\psi_4(k)$ (b).

Table 1
The first moment of the life times and number of failures.

λ	$\mathbb{E}[T_1]$	$\mathbb{E}[T_2]$	$\mathbb{E}[T_3]$	$\mathbb{E}[T_4]$	$\mathbb{E}[\Psi_{1,1}]$	$\mathbb{E}[\Psi_{1,2}]$	$\mathbb{E}[\Psi_3]$	$\mathbb{E}[\Psi_4]$
0.5	195.94	42.81	51.75	92.39	3.74	2.94	0.13	1.75
0.8	96.81	23.51	33.38	39.78	2.81	2.96	0.29	1.13
1.2	56.45	13.81	23.26	20.63	2.29	2.84	0.53	0.81
1.7	38.35	9.03	17.47	12.84	2.02	2.57	0.78	0.64

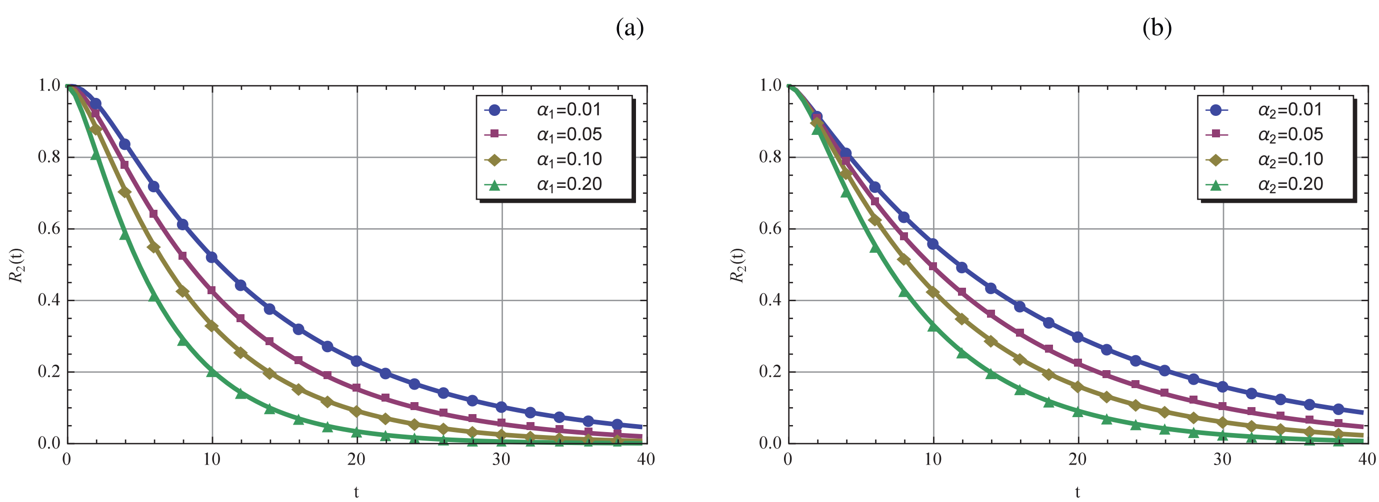

Fig. 10. The function $R_2(t)$ for α_1 (a) and α_2 (b).

respectively. To evaluate these functions, we used a numerical inversion algorithm for the corresponding Laplace transforms $\tilde{R}_n(s)$, which must be calculated in symbolic form. For the calculations we employed *Mathematica* software package from Wolfram Research. This program has some limitations regarding the volume of symbolic representations. For this reason, and in order to reduce the algorithm's evaluation time, we had to restrict the number of items of the sums in (19), (37), (46) and (57) by assuming that $q_1 = 2$ and $q_2 = 1$. As can be seen, for higher values of λ the functions exhibit heavier tails. Function $R_1(t)$ has the heaviest tail of all reliability functions evaluated. In Figs. 7a and 7b, it can be seen that, depending on the arrival rate, server 1 or server 2 may be the more reliable server at any one time. This can be explained by the threshold control policy, which regulates use of server 2. With increasing arrival rate it becomes more likely that the slower server is busy and consequently in a failed state.

In Figs. 8a, 8b, 9a and 9b, we plot the discrete probability density functions $\psi_{(0,0,0)}(k) = \mathbb{P}[\Psi_{(0,0,0)} = k]$ for the numbers of failures of server 1 and server 2 during the life time T_1, for the number of failures of server 2 during the life time T_3 and for the number of failures of server 1 during the life time T_4, respectively. The first two functions are defined on a set $\mathbb{N}$, while the other two are defined on a set $\mathbb{N}_0$. To obtain these functions, we used a numerical inversion algorithm for the corresponding PGFs. Note that, for increasing λ the probability $\psi_{(0,\,0,\,0)}(1)$ of only one failure during T_1 increases for server 1 when λ increase but decreases for server 2. The reason for this is again the threshold control policy. The same can be seen in Figs. 9a and 9b, where we observe that the probability $\psi_{(0,\,0,\,0)}(0)$ of no failure of server 2 during the time T_3 decreases, while that of server 1 during the time T_4 increases.

In Table 1, we list the moments of the life times $\mathbb{E}[T_n]$, $1 \le n \le 4$ and the number of failures $\mathbb{E}[\Psi_{1,1}]$, $\mathbb{E}[\Psi_{1,2}]$, $\mathbb{E}[\Psi_n]$, $n = 3, 4,$

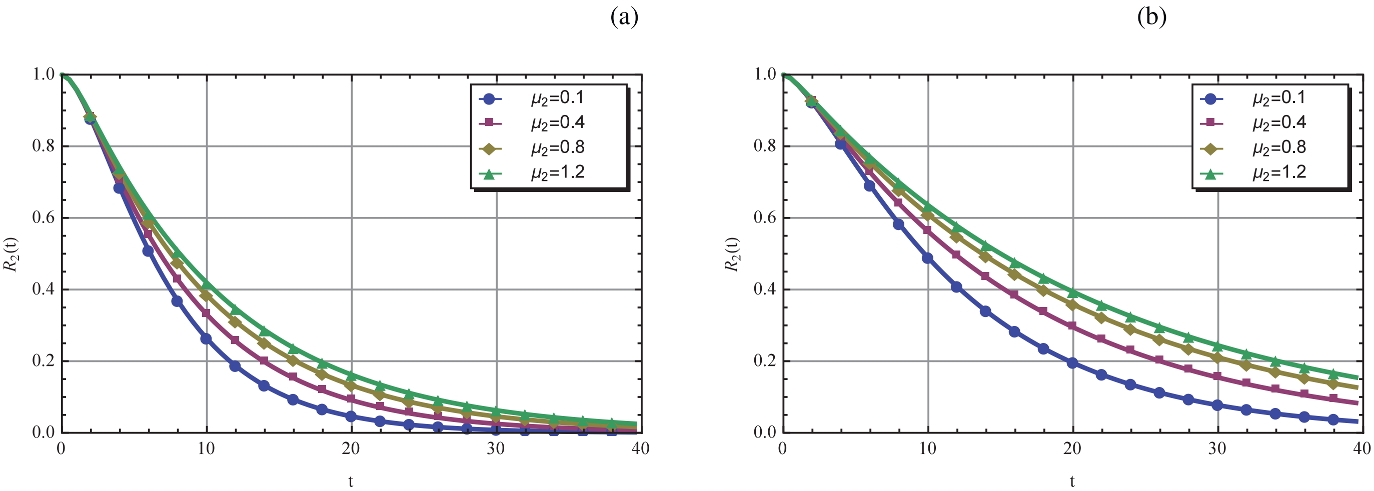

Fig. 11. The function $R_2(t)$ for $\mu_1 = 2.4$ (a) and $\mu_1 = 4.8$ (b).

during the corresponding lifetimes. As expected, the first measure is decreasing function in λ. The second measure can be different dependent on the type of the lifetime.

In Figs. 10a, 10b, 11a and 11b, we show respectively the influences of α_1, α_2, μ_1 and μ_2 on the reliability function $R_2(t)$. We observe that with decreasing of α_1 and α_2 the system becomes more reliable and the corresponding distribution functions $F_{T_2}(t) = 1 - R_2(t)$ of the life time T_2 exhibit heavier tails. It can be also noticed that the system becomes more reliable for higher values of service rates μ_1 and μ_2 due to the fact that in this case the probability for empty servers increases that in turn extends the system's life time. In this example the system is most reliable for parameters $\alpha_1 = 0.01, \alpha_2 = 0.01, \mu_1 = 4.8, \mu_2 = 1.2$.

6. Conclusion

We have provided a reliability analysis of a two-server heterogeneous unreliable queueing system with a threshold control policy for allocating customers to the servers. Our results complement the classical performance analysis of unreliable queueing models that can be described by quasi-birth-and-death processes. We used a matrix-geometric solution method to obtain the stationary state distribution and some reliability measures, such as availability and failure frequency. Combining the forward-elimination-backward-substitution method for the boundary states with the generating function approach for the states above the highest threshold level yielded a closed form solution in terms of the Laplace transform for the reliability function and, consequently, for the mean time to first failure. A distribution of the number of failures during the life time was derived, expressed in terms of the probability generating function. Finally, we presented numerical experiments to explore the effect of various system parameters on the reliability of servers and system.

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References

Legros, B., & Jouini, O. (2017). Routing in a queueing system with two heterogeneous servers in speed and in quality of resolution. *Stochastic Models, 33*, 392–410.
Bai, W. H., Xi, J. Q., Zhu, J. X., & Huang, S. W. (2015). Performance analysis of heterogeneous data centers in cloud computing using a complex queuing model. *Mathematical Problems in Engineering, 2015*.
Efrosinin, D. (2008). Controlled queueing systems with heterogeneous servers. In *Proceedings of the Dynamic optimization and monotonicity properties, saarbrücken, VDM*.
Efrosinin, D. (2013). Queueing model of a hybrid channel with faster link subject to partial and complete failures. *Annals of Operations Research, 202*, 75–102.
Efrosinin, D., Samouylov, K., & Gudkova, I. (2016). Busy period analysis of a queueing system with breakdowns and its application to wireless network under licensed shared access regime. In *Proceedings of the Internet of things, smart spaces, and next generation networks and systems* (pp. 426–439). Springer.
Efrosinin, D., & Sztrik, J. (2016). Optimal control of a two-server heterogeneous queueing system with breakdowns and constant retrials. *Information Technologies and Mathematical Modelling - Queuing Theory and Applications*, 57–72.
Gudkova, I., Samouylov, K., Ostrikova, D., Mokrov, E., Ponomorenko-Timofeev, A., Andreev, S., & Koucheryavy, Y. (2015). Service failure and interruption probability analysis for licensed shared access regulatory framework. 7th International Congress on Ultra Modern Telecommunications and Control Systems and Workshops (ICUMT), Brno 123–131.
Koole, G. (1995). A simple proof of the optimality of a threshold policy in a two-server queueing system. *Systems and Control letters, 26*, 301–303.
Kumar, B. K., & Madheswari, S. P. (2005). An $m/m/2$ queueing system with heterogeneous servers and multiple vacations. *Mathematical and Computer Modelling, 41*, 1415–1429.
Kumar, B. K., Madheswari, S. P., & Venkatakrishnan, K. S. (2007). Transient solution of an $m/m/2$ queue with heterogeneous servers subject to catastrophes. *Information and Management Sciences, 18*(1), 63–80.
Levy, J., & Yechiali, U. (1976). An $m/m/s$ queue with servers' vacations. *Information Systems and Operational Research, 14*(2), 153–163.
Lin, W., & Kumar, P. R. (1984). Optimal control of a queueing system with two heterogeneous servers. *IEEE Transactions on Automatic Control, 29*, 696–703.
Mitrany, I. L., & Avi-Itzhak, B. (1967). A many server queue with service interruptions. *Operations Research, 16*, 628–638.
Neuts, M. F. (1981). *Matrix-Geometric Solutions in Stochastic Models*. Baltimore: The John Hopkins University Press.
Neuts, M. F., & Lucantoni, D. M. (1979). A markovian queue with n servers subject to breakdowns and repairs. *Management Science, 25*, 849–861.
Ozkan, E., & Kharoufeh, J. (2014). Optimal control of a two-server queueing system with failures. *Probability in the Engineering and Informational Sciences, 28*, 489–527.
Rykov, V., & Efrosinin, D. (2009). On the slow server problem. *Automation and Remote Control, 70*(12), 2013.
Vishnevskii, V. M., Semenova, O. V., & Sharov, S. Y. (2013). Modeling and analysis of a hybrid communication channel based on free-space optical and radio-frequency technologies. *Automation and Remote Control, 74*(3), 521–528.
Yue, D., Yue, W., Yu, J., & Tian, R. (2009). A heterogeneous two-server queueing system with balking and server breakdowns. In *Proceedings of the eighth international symposium on operations research and its applications, china* (pp. 230–244).