



# Teaching Queueing Theory and its Applications

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# Outline

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- Origin of Queueing Theory
- Classifications of Queueing Systems
- Applications
- Solution Methods
- Basic Formulas and Laws
- Reputed Scientists
- Software Support
- References

# Origin of Queueing Theory

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Agner Krarup Erlang, 1878-1929

- "The Theory of Probabilities and Telephone Conversations", Nyt Tidsskrift for Matematik B, vol 20, 1909.
- "Solution of some Problems in the Theory of Probabilities of Significance in Automatic Telephone Exchanges", Elektroteknikeren, vol 13, 1917.
- "The life and works of A.K. Erlang", E. Brockmeyer, H.L. Halstrom and Arns Jensen, Copenhagen: The Copenhagen Telephone Company, 1948.

## Queueing Theory Homepage

<http://web2.uwindsor.ca/math/hlynka/queue.html>

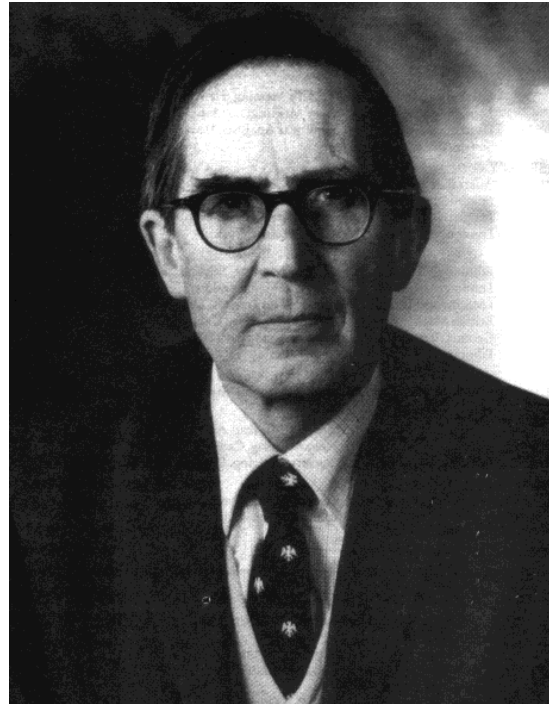
# Applications

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- Telephony, Call Centers
- Manufacturing
- Inventories
- Dams
- Supermarkets
- Computer and Communication Systems
- Sensor Networks, IoT
- Infocommunication Networks, Clouds
- Hospitals
- Many others

# Kendall's Notation

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David G. Kendall, 1918-2007

$A/B/c/K/m/Z$

# Performance Metrics

- Utilizations
- Mean Number of Customers in the System / Queue
- Mean Response / Waiting Time
- Mean Busy Period Length of the Server
- Distribution of Response / Waiting Time
- Distribution of the Busy Period
- Distribution of Number of Customers Served during a Busy Period
- Distribution of Number of Retrials until Service Completion

# Solution Methodologies

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- Analytical
- Numerical
- Asymptotic
- Simulation
- Tool Supported Solutions

# Erlang Loss Formula, M/G/c/c Systems

$$B(c, \rho) = p_c = \frac{\rho^c / c!}{\sum_{n=0}^c \rho^n / n!} \quad \rho = \lambda E(B)$$

$$B(c, \rho) = \frac{\rho B(c-1, \rho) / c}{1 + \rho B(c-1, \rho) / c} = \frac{\rho B(c-1, \rho)}{c + \rho B(c-1, \rho)}$$

$$B(0, \rho) = 1$$

# Approximation Formula

$$P_n \approx \frac{\Phi\left(\frac{n + \frac{1}{2} - \rho}{\sqrt{\rho}}\right) - \Phi\left(\frac{n - 1 + \frac{1}{2} - \rho}{\sqrt{\rho}}\right)}{\Phi\left(\frac{n + \frac{1}{2} - \rho}{\sqrt{\rho}}\right)} = 1 - \frac{\Phi\left(\frac{n - \frac{1}{2} - \rho}{\sqrt{\rho}}\right)}{\Phi\left(\frac{n + \frac{1}{2} - \rho}{\sqrt{\rho}}\right)},$$

$$\Phi(s) = \int_{-\infty}^s \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx.$$

# Pollaczek-Khintchine Formulas, M/G/1 Systems



Felix Pollachek, 1892-1981



Alexander Y. Khintchine, 1894-1959

# Mean Value Formulas

$$C_X^2 := \frac{\text{Var}[X]}{(E[X])^2}$$

$$E[W] = E[R] \frac{\rho}{1 - \rho} = \frac{E[B]}{2} \frac{\rho}{1 - \rho} (1 + C_B^2)$$

$$E[T] = E[B] \left( 1 + \frac{\rho(1 + C_B^2)}{2(1 - \rho)} \right)$$

# Transform Formulas

$$G_N(z) = L_B(\lambda(1 - z)) \cdot \frac{(1 - \rho)(1 - z)}{L_B(\lambda(1 - z)) - z}$$

$$L_T(s) = L_B(s) \frac{s(1 - \rho)}{s - \lambda + \lambda L_B(s)}$$

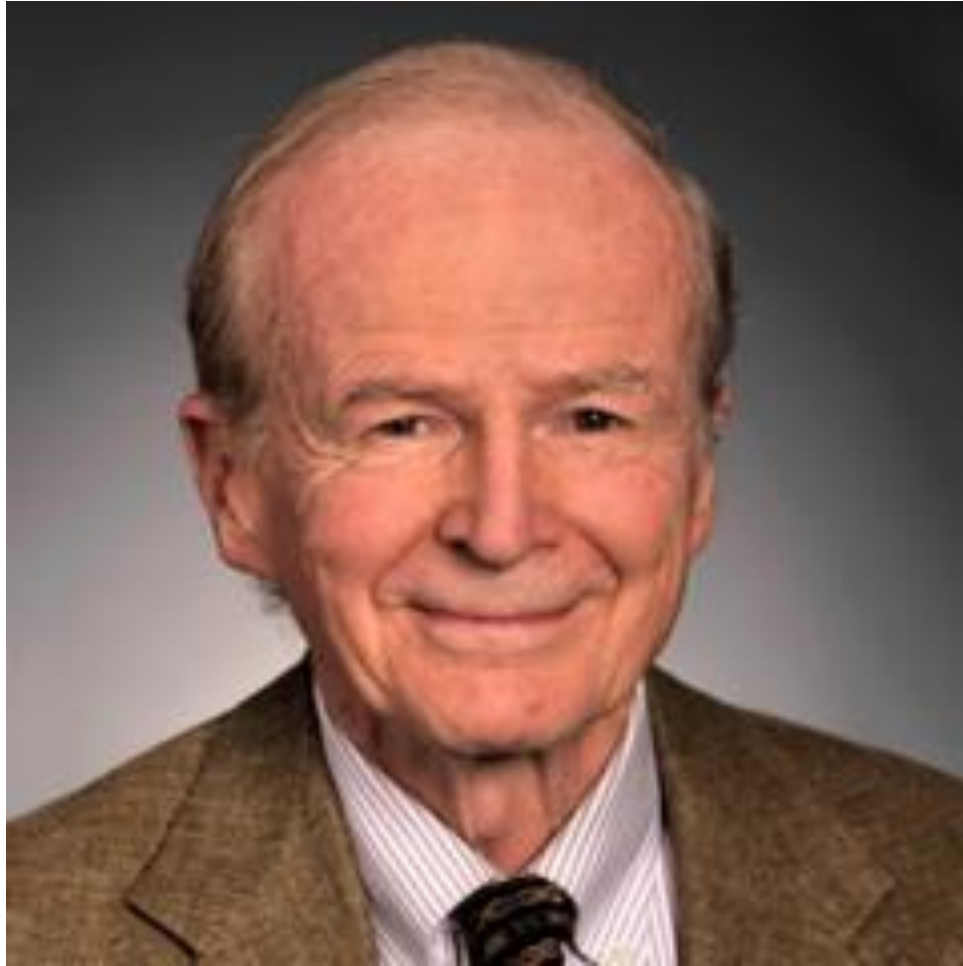
# Little's Law

$$\mathbb{E}(N) = \bar{\lambda} \mathbb{E}(T)$$

$$\mathbb{E}(Q) = \bar{\lambda} \mathbb{E}(W)$$

$$\mathbb{E}(N(N-1) \dots (N(-k+1))) = \lambda^k \mathbb{E}(T^k)$$

# John Little



John Little, 1928 -

# Boris Vladimirovich Gnedenko



Boris Vladimirovich Gnedenko, 1912-1995

# Igor Nikolaevich Kovalenko

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Igor Nikolaevich Kovalenko, 1935 - 2019

# Leonard Kleinrock



Leonard Kleinrock, 1934 -

# Lajos Takács

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Lajos Takács, 1924 -2015

# Takács Formulas, M/G/1 Systems

$$\mathbb{E}(W^k) = \frac{\lambda}{1 - \rho} \sum_{i=1}^k \binom{k}{i} \frac{\mathbb{E}(S^{i+1})}{i+1} \mathbb{E}(W^{k-i})$$

$$\mathbb{E}(T^k) = \sum_{l=0}^k \binom{n}{l} \mathbb{E}(W^l) \cdot \mathbb{E}(S^{k-l})$$

$$L_{\delta}(t) = L_S(t + \lambda - \lambda L_{\delta}(t))$$

# Tool Supported Modeling

- University of Dortmund: *HIT, HiQPN, APNN*  
<http://ls4-www.informatik.uni-dortmund.de/tools.html/>
- University of Illinois at Urbana-Champaign: *MÖBIUS*  
<http://www.mobius.uiuc.edu/>
- University of Erlangen: *PEPSY, MOSEL*  
<http://www4.informatik.uni-erlangen.de/Projects/MOSEL/>
- University of Oxford: *PRISM*  
<http://www.prismmodelchecker.org/>

# Software and Information

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<http://web2.uwindsor.ca/math/hlynka/qsoft.html>

<http://mason.gmu.edu/~jshortle/QtPlus-4-0.zip>

## **QSA ( Queueing Systems Assistance)**

<https://qsa.inf.unideb.hu>

## **Lecture Notes**

[https://irh.inf.unideb.hu/~jsztrik/education/16/SOR\\_Main\\_Angol.pdf](https://irh.inf.unideb.hu/~jsztrik/education/16/SOR_Main_Angol.pdf)

[https://irh.inf.unideb.hu/~jsztrik/education/16/Queueing\\_Problems\\_Solutions\\_2021\\_Sztrik.pdf](https://irh.inf.unideb.hu/~jsztrik/education/16/Queueing_Problems_Solutions_2021_Sztrik.pdf)

# Introduction of QSA and Case Studies

## Example 1

Customers arrive to a 2 server system according to a Poisson process with rate 3. The service times are exponentially distributed with parameter 2.

**Find** the minimum capacity of the system for which the probability of blocking is less than 0.01 and the probability that the waiting time exceeds 1.8 minutes is less than 0.05.

# Case Studies

## Example 2

We have a finite-source system with 50 sources, the request generation times are exponentially distributed with rate 0.5. The service times are exponentially distributed for all the 5 servers with intensity 2.

**Find** the minimum capacity of the system for which the probability of blocking is less than 0.01 and the probability that the waiting time exceeds 3.5 minutes is less than 0.05.







# Case Studies

## Example 3






Let us see an M/M/1 system with arrival intensity 1 and the following costs, cost of service per server per unit time  $C_S = 2$ , cost of waiting in the system per customer per unit time  $C_W = 2$ , cost of idleness per server per unit time  $C_I = 10$ , cost of service rate per server per unit time  $C_{SR} = 10$ , reward per customer per unit time  $R = 5$ .

**Find** the optimal value for the service intensity which minimize the expected total cost per unit time with linear objective function.






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for Your  
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