Asymptotic Sojourn Time Analysis of Finite-Source M/M/1 Retrial Queuing System with Two-Way Communication

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Abstract. The aim of the present paper is to investigate a retrial queuing system *M/M/1* with a finite number of sources and two-way communication. Each source can generate a request after an exponentially distributed time and will not generate another one until the previous call return to the source. If an incoming customer finds the server idle its service starts. Otherwise, if the server is busy an arriving (primary or repeated) customer moves into the orbit and after some exponentially distributed time it retries to enter the server. When the server is idle it generates an outgoing call after an exponentially distributed time with different parameters to the customers in the orbit and to the sources, respectively. The service times of the incoming and outgoing calls are exponentially distributed with different rates. Applying method of asymptotic analysis under the condition of unlimited growing number of sources it is proved that the limiting sojourn/waiting time of the customer in the system follows a generalized exponential distribution with given parameters. In addition, the asymptotic average number of customers in the orbit is obtained.

Keywords: Finite-source queuing system · Retrial queues
Call centers · Two-way communication · Asymptotic analysis
Sojourn time distribution

1 Introduction

Modeling retrial queueing systems with two-way communication has been becoming more and more popular topic of investigations for the last years. The main reason is that in many applications, for example in call centers where the agents could make outgoing calls to advertise, promote and sell packages and services of the center, it is important to increase the utilization the server, see for example [1,2,5,10,17,20]. The main feature of two-way communication is that and idle server can generate outgoing calls to the source (primary calls) or

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to the orbit (retrial calls). If a primary outgoing call finds the server busy it is lost in infinite source case or returns to the source in finite-source case. If at the arrival of a retrial outgoing request the server is busy it goes back to the orbit and can generate a retrial call. The first results on infinite source retrial queueing systems with two-way communication was published by Falin [9], followed by some recent ones, see for example [3,4,6,7,13,14,16,18,19].

Finite-source retrial queueing systems with two-way communication has not been investigated intensively, yet. To the best knowledge of the authors only the paper of Dragieva and Phung-Duc [8] dealt with this problem. They investigated an M/M/1//N retrial model with exponentially distributed retrial times where the primary and retrial outgoing call generation and service times are also exponentially distributed. Recursive formulas for computing the steady-state distribution of the system state were derived as well as expressions for the main performance macro characteristics in terms of the server utilization were obtained. Numerical examples were presented. It is easy to see that by choosing parameters in an appropriate way the previous results for one-way communication and as limiting case for infinite source models two-way communication can be obtained.

In their Conclusion the authors mentioned studying waiting time process among others. Hence it was our main motivation to investigate the distribution of the waiting and response time distribution of primary incoming calls by using asymptotic methods similar to [11,12,15]. Assuming that the number of sources N tends to infinity it is proofed that the response/waiting time distribution of primary incoming customers in the system/orbit can be approximated by a generalized exponential distribution with given parameters. In addition, the asymptotic average number of customers in the system and in the orbit are obtained. The results are validated by the Little-formula.

The rest of the paper is organized as follows. In Sect. 2 description of the model is given, the corresponding 2-dimensional Markov process is defined. In Sect. 3 the mean normalized number customers in the orbit is obtained. Section 4 deals with the distribution of the response and waiting time of calls. Finally, the paper ends with a Conclusion and some future plans are highlighted.

2 Model Description and Notations

Let us consider a retrial queuing system of type M/M/1//N with two-way communication. The number of sources is N and each of them can generate a primary request with rate λ/N. A source cannot generate a new call until the end of the successful service of this customer. If incoming (primary or retrial) customer finds the server idle, it enters into service immediately, in which the required service time is exponentially distributed random variable with parameter μ_1. Otherwise, if the server is busy, an arriving (primary or repeated) customer moves into the orbit. The retrial times of requests are assumed to be exponentially distributed with rate σ/N. We suppose that if the server is idle, it generates an outgoing call in an exponentially distributed time with rate α/N for outgoing call to the orbit

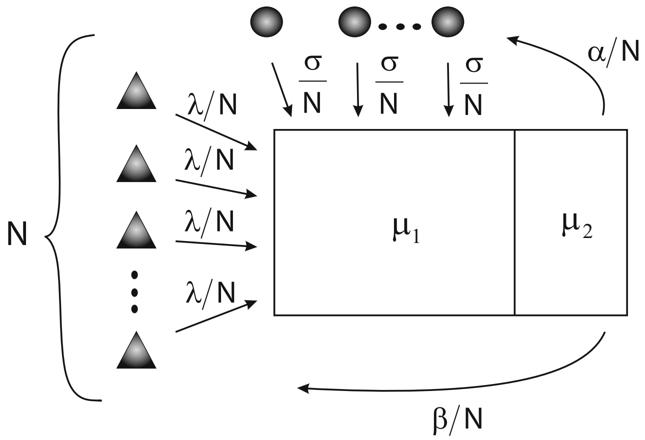

Fig. 1. Retrial queuing system of type M/M/1//N with two-way communication

and with rate β/N for primary outgoing calls. The service times of outgoing calls are assumed to be exponentially distributed random variable with parameter μ_2. All random variables involved in the model construction are assumed to be independent of each other.

Our main aim is to find the sojourn time distribution of the customers in the system and in the orbit, respectively. The method of asymptotic analysis is used in the condition of an unlimited growing number of sources.

First, we will find the first order asymptotic mean normed number of customers in the orbit, the results of which we will apply later on to study the sojourn time distribution of the customer in the system.

3 First Order Asymptotic for the Number of Customers in the Orbit

Let $Q(t)$ be the number of customers on the orbit at time t, $C(t)$ be the server state at time t, that is

$$C(t) = \begin{cases} 0, & \text{if the server is idle,} \\ 1, & \text{if the server is busy by an incoming call,} \\ 2, & \text{if the server is busy by an outgoing call.} \end{cases}$$

Thus, we will investigate the Markov process $\{C(t), Q(t)\}$.

Let us define the stationary probabilities as follows:

$$P_k(n) = \lim_{t\to\infty} P\{C(t) = k, Q(t) = n\}.$$

For the stationary probability distribution $P_k(n)$ by using standard methods we can obtain the following system of Kolmogorov equations, namely

$$\begin{aligned}
&-\left[\lambda+\beta+(\sigma+\alpha-\lambda-\beta)\frac{n}{N}\right]P_0(n)+\mu_1 P_1(n)+\mu_2 P_2(n)=0,\\
&-\left[\lambda+\mu_1-\lambda\frac{n+1}{N}\right]P_1(n)+\lambda\left(1-\frac{n}{N}\right)[P_0(n)+P_1(n-1)]\\
&+\sigma\frac{n+1}{N}P_0(n+1)=0,\\
&-\left[\lambda+\mu_2-\lambda\frac{n+1}{N}\right]P_2(n)+\beta\left(1-\frac{n}{N}\right)P_0(n)+\alpha\frac{n+1}{N}P_0(n+1)\\
&+\lambda\left(1-\frac{n}{N}\right)P_2(n-1)=0.
\end{aligned} \tag{1}$$

In paper [8] although the notations are different basically this system was solved by using a recursive numerical algorithm. Since our aim is to get the sojourn time distribution of the customers we follow an asymptotic method because to obtain the exact distribution we need rather complicated approach. Let us denote the partial characteristic functions as

$$H_k(u)=\sum_{n=0}^{N}e^{iun}P_k(n),$$

where $i=\sqrt{-1}$ is imaginary unit, then system (1) will be rewritten in the form

$$\begin{aligned}
&-(\lambda+\beta)H_0(u)+\mu_1 H_1(u)+\mu_2 H_2(u)+i\frac{[\sigma+\alpha-\lambda-\beta]}{N}H_0'(u)=0,\\
&\lambda H_0(u)+\left[\lambda\left(e^{iu}-1\right)\left(1-\frac{1}{N}\right)-\mu_1\right]H_1(u)\\
&+i\frac{\left(\lambda-\sigma e^{-iu}\right)}{N}H_0'(u)+i\frac{\lambda\left(e^{iu}-1\right)}{N}H_1'(u)=0,\\
&\beta H_0(u)+\left(\lambda\left(e^{iu}-1\right)\left(1-\frac{1}{N}\right)-\mu_2\right)H_2(u)\\
&+i\frac{\left(\beta-\alpha e^{-iu}\right)}{N}H_0'(u)+i\frac{\lambda\left(e^{iu}-1\right)}{N}H_2'(u)=0.
\end{aligned} \tag{2}$$

Summarizing equations of system (2) we receive an additional equality of the form

$$\begin{aligned}
&\lambda\left(e^{iu}-1\right)\left(1-\frac{1}{N}\right)[H_1(u)+H_2(u)]+i\frac{(\alpha+\sigma)\left(1-e^{-iu}\right)}{N}H_0'(u)\\
&+i\frac{\lambda\left(e^{iu}-1\right)}{N}\left[H_1'(u)+H_2'(u)\right]=0.
\end{aligned} \tag{3}$$

We will solve system (2) and (3) by the method of asymptotic analysis in the condition of an infinitely increasing number of sources $N\to\infty$.

Theorem 1. *Let Q be the number of customers in the orbit then*

$$\lim_{N\to\infty} E\exp\left\{iw\frac{Q}{N}\right\} = \exp\left\{iw\kappa\right\}, \tag{4}$$

where value of parameter κ is the positive solution of the equation

$$\lambda(1-\kappa)\left[R_1(\kappa)+R_2(\kappa)\right]-(\alpha+\sigma)R_0(\kappa)\kappa = 0. \tag{5}$$

Here the stationary distributions of probabilities $R_k(\kappa)$ of the service state k depends on κ and can be obtained as follows

$$\begin{gathered} R_0(\kappa) = \left\{1+\frac{1}{\mu_1}\left[\lambda(1-\kappa)+\sigma\kappa\right]+\frac{1}{\mu_2}\left[\beta(1-\kappa)+\alpha\kappa\right]\right\}^{-1}, \\ R_1(\kappa) = \frac{1}{\mu_1}\left[\lambda(1-\kappa)+\sigma\kappa\right]R_0(\kappa), \\ R_2(\kappa) = \frac{1}{\mu_2}\left[\beta(1-\kappa)+\alpha\kappa\right]R_0(\kappa). \end{gathered} \tag{6}$$

Proof. Designating $\frac{1}{N} = \varepsilon$, in systems (2–3) let us introduce the following replacements

$$u = \varepsilon w, \qquad H_k(u) = F_k(w,\varepsilon), \tag{7}$$

then systems (2–3) can be rewritten as

$$\begin{aligned} & -(\lambda+\beta)F_0(w,\varepsilon)+\mu_1F_1(w,\varepsilon)+\mu_2F_2(w,\varepsilon) \\ & +i\left[\sigma+\alpha-\lambda-\beta\right]\tfrac{\partial F_0(w,\varepsilon)}{\partial w} = 0, \\ & \lambda F_0(w,\varepsilon)+\left[\lambda\left(e^{i\varepsilon w}-1\right)(1-\varepsilon)-\mu_1\right]F_1(w,\varepsilon) \\ & +i\left(\lambda-\sigma e^{-i\varepsilon w}\right)\tfrac{\partial F_0(w,\varepsilon)}{\partial w}+i\lambda\left(e^{i\varepsilon w}-1\right)\tfrac{\partial F_1(w,\varepsilon)}{\partial w} = 0, \\ & \beta F_0(w,\varepsilon)+\left[\lambda\left(e^{i\varepsilon w}-1\right)(1-\varepsilon)-\mu_2\right]F_2(w,\varepsilon) \\ & +i\left(\beta-\alpha e^{-i\varepsilon w}\right)\tfrac{\partial F_0(w,\varepsilon)}{\partial w}+i\lambda\left(e^{i\varepsilon w}-1\right)\tfrac{\partial F_2(w,\varepsilon)}{\partial w} = 0, \\ & \lambda(1-\varepsilon)\left[F_1(w,\varepsilon)+F_2(w,\varepsilon)\right]+i(\alpha+\sigma)e^{-i\varepsilon w}\frac{\partial F_0(w,\varepsilon)}{\partial w} \\ & +i\lambda\left[\tfrac{\partial F_1(w,\varepsilon)}{\partial w}+\tfrac{\partial F_2(w,\varepsilon)}{\partial w}\right] = 0. \end{aligned} \tag{8}$$

Denoting $\lim_{\varepsilon \to 0} F_k(w, \varepsilon) = F_k(w)$, let us execute this limiting transition in system (8) and as result we will obtain

$$\begin{aligned}
&-(\lambda + \beta) F_0(w) + \mu_1 F_1(w) + \mu_2 F_2(w) + i [\sigma + \alpha - \lambda - \beta] F_0^{'}(w) = 0, \\
&\lambda F_0(w) - \mu_1 F_1(w) + i (\lambda - \sigma) F_0^{'}(w) = 0, \\
&\beta F_0(w) - \mu_2 F_2(w) + i (\beta - \alpha) F_0^{'}(w) = 0, \\
&\lambda [F_1(w) + F_2(w)] + i(\alpha + \sigma) F_0^{'}(w) + i\lambda \left[F_1^{'}(w) + F_2^{'}(w)\right] = 0.
\end{aligned} \tag{9}$$

We show that the solution of the system (9) can be written in the following form

$$F_k(w) = R_k \Phi(w), \tag{10}$$

where R_k the limiting probability distributions of the service state k under conditions $N \to \infty$ and $\Phi(w)$ is the limiting characteristic function of the normalized number of customers in the orbit. Substituting solution (10) in (9) we obtain

$$\begin{aligned}
&-(\lambda + \beta) R_0 + \mu_1 R_1 + \mu_2 R_2 + i [\sigma + \alpha - \lambda - \beta] R_0 \frac{\Phi^{'}(w)}{\Phi(w)} = 0, \\
&\lambda R_0 - \mu_1 R_1 + i (\lambda - \sigma) R_0 \frac{\Phi^{'}(w)}{\Phi(w)} = 0, \\
&\beta R_0 - \mu_2 R_2 + i (\beta - \alpha) R_0 \frac{\Phi^{'}(w)}{\Phi(w)} = 0, \\
&\lambda [R_1 + R_2] + i \{(\alpha + \sigma) R_0 + \lambda (R_1 + R_2)\} \frac{\Phi^{'}(w)}{\Phi(w)} = 0.
\end{aligned} \tag{11}$$

From the form of equations of system (11) it follows that quantity $\frac{\Phi^{'}(w)}{\Phi(w)}$ does not depend on w, then we can conclude that function $\Phi(w)$ has a form

$$\Phi(w) = \exp(iw\kappa), \tag{12}$$

coinciding with equality (4).

Now, to find κ let us substitute the explicit form of the function $\Phi(w)$ in the equations of system (11) and we obtain the following system

$$\begin{aligned}
&-(\lambda + \beta) R_0 + \mu_1 R_1 + \mu_2 R_2 - [\sigma + \alpha - \lambda - \beta] R_0 \kappa = 0, \\
&\lambda R_0 - \mu_1 R_1 - (\lambda - \sigma) R_0 \kappa = 0, \\
&\beta R_0 - \mu_2 R_2 - (\beta - \alpha) R_0 \kappa = 0, \\
&\lambda (1 - \kappa) [R_1 + R_2] - (\alpha + \sigma) R_0 \kappa = 0.
\end{aligned} \tag{13}$$

The quantity κ is the solution of the fourth equation of system (13), which coincides with (5) showing a probabilistic interpretation that the mean arrival rate to the orbit is equal to the mean departure rate from the orbit. From the second and third equations of system (13) taking into account the normalization condition, it is not difficult to obtain expressions for the quantities R_0, R_1 and R_2. Let us note that since R_k is the solution of system (13) whose coefficients depend on κ, then $R_k = R_k(\kappa)$ and their form coincides with (6).

The theorem is proved. □

4 Sojourn Time Distribution of the Customer in the System

Let T be the total sojourn time of the tagged customer in the system and $T(t)$ is the time length from moment t until the end of the service of the tagged customer. The total sojourn time T is simply expressed through the residual sojourn time $T(t)$.

Let $S(t)$ describe the server state at time t as follows

$$S(t) = \begin{cases} 0, & \text{server is free,} \\ 1, & \text{server is busy by incoming (not tagged) customer,} \\ 2, & \text{server is busy by outgoing (not tagged) customer,} \\ 3, & \text{server is busy by incoming tagged customer,} \\ 4, & \text{server is busy by outgoing tagged customer.} \end{cases}$$

We will define the conditional characteristic functions in the form

$$G_k(u,n,t) = \mathsf{E}\left\{e^{iuT(t)}|S(t)=k, Q(t)=n\right\}.$$

Assuming that the system is operating in steady-state, it is not difficult to write the following system of Kolmogorov equations

$$\begin{aligned}
&\left[iu - (\lambda+\beta)\frac{N-n}{N} - (\sigma+\alpha)\frac{n}{N}\right]G_0(u,n) + \lambda\frac{N-n}{N}G_1(u,n) \\
&+\sigma\tfrac{n-1}{N}G_1(u,n-1) + \beta\tfrac{N-n}{N}G_2(u,n) + \alpha\tfrac{n-1}{N}G_2(u,n-1) \\
&+\tfrac{\sigma}{N}G_3(u,n-1) + \tfrac{\alpha}{N}G_4(u,n-1) = 0, \\
&\left[iu - \lambda\frac{N-n-1}{N} - \mu_1\right]G_1(u,n) + \lambda\frac{N-n-1}{N}G_1(u,n+1) \\
&+\mu_1 G_0(u,n) = 0, \\
&\left[iu - \lambda\frac{N-n-1}{N} - \mu_2\right]G_2(u,n) + \lambda\frac{N-n-1}{N}G_2(u,n+1) \\
&+\mu_2 G_0(u,n) = 0, \\
&\left[iu - \lambda\frac{N-n-1}{N} - \mu_1\right]G_3(u,n) + \lambda\frac{N-n-1}{N}G_3(u,n+1) + \mu_1 = 0, \\
&\left[iu - \lambda\frac{N-n-1}{N} - \mu_2\right]G_4(u,n) + \lambda\frac{N-n-1}{N}G_4(u,n+1) + \mu_2 = 0.
\end{aligned} \tag{14}$$

The method of asymptotic analysis, see [11] is applied to prove the following theorem

Theorem 2. *Let T be the total sojourn time of the customer in the system then*

$$\lim_{N\to\infty} \mathsf{E}\exp\left\{iw\frac{T}{N}\right\} = R_0 + (1-R_0)\frac{(\alpha+\sigma)\,R_0}{(\alpha+\sigma)\,R_0 - iw}. \tag{15}$$

Proof. Let us denote $\frac{1}{N} = \varepsilon$. Executing the following substitutions in system (14)

$$u = \varepsilon w, \quad \varepsilon n = x, \quad G_k(u,n) = g_k(w,x,\varepsilon), \tag{16}$$

we obtain this system in the following form

$$\begin{aligned}
&[i\varepsilon w - (\lambda+\beta)(1-x) - (\sigma+\alpha)\,x]\,g_0(w,x,\varepsilon) + \lambda(1-x)g_1(w,x,\varepsilon)\\
&+\sigma(x-\varepsilon)g_1(w,x-\varepsilon,\varepsilon) + \beta(1-x)g_2(w,x,\varepsilon)\\
&+\alpha(x-\varepsilon)g_2(w,x-\varepsilon,\varepsilon) + \sigma\varepsilon g_3(w,x-\varepsilon,\varepsilon) + \alpha\varepsilon g_4(w,x-\varepsilon,\varepsilon) = 0,\\
&[i\varepsilon w - \lambda(1-x-\varepsilon) - \mu_1]\,g_1(w,x,\varepsilon) + \lambda(1-x-\varepsilon)g_1(w,x+\varepsilon,\varepsilon)\\
&+\mu_1 g_0(w,x,\varepsilon) = 0,\\
&[i\varepsilon w - \lambda(1-x-\varepsilon) - \mu_2]\,g_2(w,x,\varepsilon) + \lambda(1-x-\varepsilon)g_2(w,x+\varepsilon,\varepsilon)\\
&+\mu_2 g_0(w,x,\varepsilon) = 0,\\
&[i\varepsilon w - \lambda(1-x-\varepsilon) - \mu_1]\,g_3(w,x,\varepsilon) + \lambda(1-x-\varepsilon)g_3(w,x+\varepsilon,\varepsilon)\\
&+\mu_1 = 0,\\
&[i\varepsilon w - \lambda(1-x-\varepsilon) - \mu_2]\,g_4(w,x,\varepsilon) + \lambda(1-x-\varepsilon)g_4(w,x+\varepsilon,\varepsilon)\\
&+\mu_2 = 0.
\end{aligned} \tag{17}$$

Denoting $\lim_{\varepsilon\to 0} g_k(w,x,\varepsilon) = g_k(w,x)$, we carry out limiting transition under condition $\varepsilon \to 0$ in the system (17) and the we get

$$\begin{aligned}
&-[(\lambda+\beta)(1-x) + (\sigma+\alpha)\,x]\,g_0(w,x) + [\lambda(1-x) + \sigma x]\,g_1(w,x)\\
&+[\beta(1-x) + \alpha x]\,g_2(w,x) = 0,\\
&\mu_1\,[g_0(w,x) - g_1(w,x)] = 0,\\
&\mu_2\,[g_0(w,x) - g_2(w,x)] = 0,\\
&\mu_1\,[1 - g_3(w,x)] = 0,\\
&\mu_2\,[1 - g_4(w,x)] = 0.
\end{aligned} \tag{18}$$

From the obtained system it follows that functions $g_3(w,x)$ and $g_4(w,x)$ are equal to unity, and functions $g_0(w,x)$, $g_1(w,x)$ and $g_2(w,x)$ are coincide.

Designating by $g(w,x)$ their common value we can write

$$g(w,x) = g_0(w,x) = g_1(w,x) = g_2(w,x).$$

Thus, the solution of the system (17) can be written in the form of decomposition

$$g_k(w,x,\varepsilon) = g(w,x) + \varepsilon f_k(w,x) + O(\varepsilon^2), \quad k = \overline{0,2},$$

which we substitute into the first three equations of the system (17) and as a result we obtain

$$\begin{aligned}
&[i\varepsilon w - (\lambda+\beta)(1-x) - (\sigma+\alpha)x]\, g(w,x) + \sigma\varepsilon g_3(w,x) + \alpha\varepsilon g_4(w,x) \\
&-\varepsilon\left[(\lambda+\beta)(1-x) + (\sigma+\alpha)x\right] f_0(w,x) + \varepsilon\left[\lambda(1-x) + \sigma x\right] f_1(w,x) \\
&+\left[\lambda(1-x) + \sigma(x-\varepsilon)\right] g(w,x) + \left[\beta(1-x) + \alpha(x-\varepsilon)\right] g(w,x) \\
&+\varepsilon\left[\beta(1-x) + \alpha x\right] f_2(w,x) - \varepsilon(\sigma+\alpha)x\tfrac{\partial g(w,x)}{\partial x} = O(\varepsilon^2), \\
&[i\varepsilon w - \mu_1]\, g(w,x) - \varepsilon\mu_1 f_1(w,x) + \mu_1 g(w,x) + \varepsilon\mu_1 f_0(w,x) \\
&+\varepsilon\lambda(1-x)\frac{\partial g(w,x)}{\partial x} = O(\varepsilon^2), \\
&[i\varepsilon w - \mu_2]\, g(w,x) - \varepsilon\mu_2 f_2(w,x) + \mu_2 g(w,x) + \varepsilon\mu_2 f_0(w,x) \\
&+\varepsilon\lambda(1-x)\frac{\partial g(w,x)}{\partial x} = O(\varepsilon^2).
\end{aligned} \tag{19}$$

After performing simple transformations in system (19) and taking into account that $g_3(w,x) = g_4(w,x) = 1$ we get

$$\begin{aligned}
&-\left[(\lambda+\beta)(1-x) + (\sigma+\alpha)x\right] f_0(w,x) + \left[\lambda(1-x) + \sigma x\right] f_1(w,x) \\
&+\left[\beta(1-x) + \alpha x\right] f_2(w,x) = \left[\sigma + \alpha - iw\right] g(w,x) - (\sigma+\alpha) \\
&+(\sigma+\alpha)x\tfrac{\partial g(w,x)}{\partial x} = 0, \\
&\mu_1\left[f_0(w,x) - f_1(w,x)\right] = -iwg(w,x) - \lambda(1-x)\frac{\partial g(w,x)}{\partial x} = 0, \\
&\mu_2\left[f_0(w,x) - f_2(w,x)\right] = -iwg(w,x) - \lambda(1-x)\frac{\partial g(w,x)}{\partial x} = 0.
\end{aligned} \tag{20}$$

Multiplying the first equation of system (20) by R_0, the second equation by R_1, the third by R_2 and then adding them we receive the following equality

$$\begin{aligned}
&\left[-\left(\lambda+\beta\right)R_0-\left(\sigma+\alpha-\lambda-\beta\right)R_0x+\mu_1R_1+\mu_2R_2\right]f_0(w,x)\\
&+\left[\lambda R_0-(\lambda-\sigma)R_0x-\mu_1R_1\right]f_1(w,x)\\
&+\left[\beta R_0-(\beta-\alpha)R_0x-\mu_2R_2\right]f_2(w,x)=\left[(\sigma+\alpha)R_0-iw\right]g(w,x)\\
&-(\sigma+\alpha)R_0+\left\{(\sigma+\alpha)R_0x-\lambda(1-x)(R_1+R_2)\right\}\tfrac{\partial g(w,x)}{\partial x}=0.
\end{aligned} \tag{21}$$

From Theorem 1 it follows that $\dfrac{n}{N}=\kappa$. By virtue of the substitutions (16) carried out earlier, namely $x=n\varepsilon$ we can conclude that $x=\kappa$.

Let us perform the substitution $x=\kappa$ in the equations of system (21) and taking into account system (13), we can find that the multipliers for functions $f_0(w,x)$, $f_1(w,x)$, $f_2(w,x)$ and $\dfrac{\partial g(w,x)}{\partial x}$ are equal to zero.

As a result equality (21) can be rewritten in the form

$$\left[(\sigma+\alpha)R_0-iw\right]g(w)=(\sigma+\alpha)R_0,$$

from which it obviously follows that

$$g(w)=\frac{(\sigma+\alpha)R_0}{(\sigma+\alpha)R_0-iw}. \tag{22}$$

Thus we obtain the characteristic function of the probability distribution of the normalized residual sojourn time *T(t)/N*. The using the law of total probability we can receive the characteristic function of the probability distribution of the normalized total sojourn time T/N of the customers in the system in the following form

$$\begin{aligned}
&\lim_{N\to\infty}\mathsf{E}\exp\left\{iw\frac{T}{N}\right\}=R_0g_3(w)+R_1g_1(w)+R_2g_2(w)\\
&=R_0+(R_1+R_2)g(w)=R_0+(1-R_0)\tfrac{(\sigma+\alpha)R_0}{(\sigma+\alpha)R_0-iw},
\end{aligned} \tag{23}$$

which coincides with (15).

The theorem is proved. □

In addition let us perform in Eq. (23) reverse replacement $w=\dfrac{u}{\varepsilon}=uN$ and denoting by $q=1-R_0$ and $\gamma=(\sigma+\alpha)R_0/N$ we receive

$$\mathsf{E}\exp\left\{iuT\right\}\approx R_0+(1-R_0)\frac{(\sigma+\alpha)R_0/N}{(\sigma+\alpha)R_0/N-iu}=1-q+q\frac{\gamma}{\gamma-iu},$$

which is the prelimit value, that is for fixed N.

Thus we obtain that the sojourn time of the customer in the system follows a generalized exponential distribution with parameters q and γ.

Consequently the mean response time can be approximated by $\frac{(1-R_0)}{(\sigma+\alpha)R_0/N}$.

Since the service time of a primary incoming customer is bounded then the limiting distribution of the normalized response time and the waiting time is coincide. Similarly, the limiting distribution of the normalized number of customers in the system and in the orbit are the same.

Hence the mean arrival rate to the system is $\lambda(1-\kappa)$.

We will use the Little-formula to check our results, namely we have

$$\lambda(1-\kappa)\frac{(1-R_0)}{(\sigma+\alpha)R_0} = \kappa$$

which is equivalent to Eq. 5 from which κ was determined.

5 Conclusion and Future Work

In this paper a finite-source retrial queuing system of type *M/M/1//N* with two-way communication was considered. The research has been performed by the method of asymptotic analysis under the condition of unlimited growing number of sources. As the result of the analysis it was shown that the limiting sojourn/waiting time of the customer in the system has a generalized exponential distribution with given parameters. The authors plan to continue their research, among others modeling finite-source retrial queuing systems with two-way communication for the case of generally distributed service times.

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