

MODELLING OF A SINGLE BUS MULTIPROCESSOR SYSTEM OPERATING IN MARKOVIAN ENVIRONMENTS

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(Received March 1991)

Abstract—In this paper, performance measures of a multiprocessor computer system with a single bus are studied by a queueing theoretic approach. The processors and the shared bus are assumed to operate in independent Markovian environments. The time intervals from the completion of the previous bus usage to the generation of a new request for each processor as well as the times that a processor uses the bus as the result of arbitration are supposed to be exponentially distributed random variables with parameter depending on the state of the corresponding environmental process. Supposing that the arrival rate of the requests is much greater than their service rate ("fast" arrival), it is shown that the busy period length of the bus converges weakly, under appropriate norming, to an exponentially distributed random variable. As a utilization, the throughput, the mean delay time, the expected waiting time, the average number of requests served during a busy period, and the mean number of active processors are calculated. Moreover, exact and approximate validation results are presented to illustrate the credibility of the proposed method.

1. INTRODUCTION

In multiprocessor systems, the contention for a common bus is one of the major factors affecting the computer performance. Several papers have been devoted to the analysis of such systems under different conditions on access rates, the distribution function of holding times, and bus arbitration protocols [1-3]. More recently Ishigaki [4] suggested a queueing theoretic approach to analyze the system and a numerical technique was used for the evaluation of the basic performance measures. In this work, an asymptotic queueing theoretic approach is proposed to study the performance of a homogeneous multiprocessor system evolving in random environments. All random times in the system are considered to be exponentially distributed, the access and service rates depend on the state of the corresponding random environments. Under a heavy traffic assumption (i.e., "fast" arrivals), it is shown that the busy period length of the bus converges weakly, under appropriate norming, to an exponentially distributed random variable. This result facilitates the calculation of several steady-state performance measures of interest.

Note that the asymptotic technique has a widespread applicability in the field of reliability theory [5-8]. Refinements in the model are often needed when the system environment is subject to randomly occurring fluctuations which appear as changes in the parameters of the model. The fluctuations may be due to changes in the physical environment, personnel changes, alteration of computer system usage intensity, etc., [9-13]. Preliminary theoretical results are introduced in Section 2. The queueing model and the proposed asymptotic approach are presented in Section 3. The derivation of the main performance measures of interest is described in Section 4. In Section 5, some numerical results illustrate the problem in question.

Research done while J. Sztrik was visiting the Department of Mathematics, University of Bradford, England and supported by István Széchenyi Foundation, Hungary and the National Foundation for Scientific Research under Grant OTKA-1648/1991, Hungary.

The author is very grateful to Professor Hideaki Takagi for providing him with the paper by Ishigaki *et al.* [4].

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2. PRELIMINARY RESULTS

This section presents a brief survey of results [5] to be applied in the next section.

Let $(X_\varepsilon(k), k \geq 0)$ be a Markov chain with state space

$$\bigcup_{q=0}^{m+1} X_q, \quad X_i \cap X_j = \emptyset, \quad i \neq j,$$

with $m+2$ levels of states, $i, j = 0, 1, \dots, m+1$, defined by the transition matrix $(p_\varepsilon(i^{(q)}, j^{(z)}))$, $i^{(q)} \in X_q, j^{(z)} \in X_z, q, z = 0, 1, \dots, m+1$ satisfying the following conditions:

- (1) $p_\varepsilon(i^{(0)}, j^{(0)}) \rightarrow p_0(i^{(0)}, j^{(0)})$, as $\varepsilon \rightarrow 0$, $i^{(0)}, j^{(0)} \in X_0$, and matrix $P_0 = (p_0(i^{(0)}, j^{(0)}))$ is irreducible;
- (2) $p_\varepsilon(i^{(q)}, j^{(q+1)}) = \varepsilon \alpha^{(q)}(i^{(q)}, j^{(q+1)}) + o(\varepsilon)$, $i^{(q)} \in X_q, j^{(q+1)} \in X_{q+1}$, where $\alpha^{(q)}(i^{(q)}, j^{(q+1)})$ is an appropriate transition matrix;
- (3) $p_\varepsilon(i^{(q)}, f^{(q)}) \rightarrow 0$, as $\varepsilon \rightarrow 0$, $i^{(q)}, f^{(q)} \in X_q, q \geq 1$;
- (4) $p_\varepsilon(i^{(q)}, f^{(z)}) \equiv 0$, $i^{(q)} \in X_q, f^{(z)} \in X_z, z - q \geq 2$.

In the sequel the set of states X_q is called the q^{th} level of the chain, $q = 0, \dots, m+1$. Let us single out the subset of states

$$\langle \alpha_m \rangle = \bigcup_{q=0}^m X_q.$$

Denote by $\{\pi_\varepsilon(i^{(q)}), i^{(q)} \in X_q\}$, $q = 1, \dots, m$ the stationary distribution of a chain with transition matrix

$$\left(\frac{p_\varepsilon(i^{(q)}, j^{(z)})}{1 - \sum_{k^{(m+1)} \in X_{m+1}} p_\varepsilon(i^{(q)}, k^{(m+1)})} \right), \quad i^{(q)} \in X_q, j^{(z)} \in X_z, q, z \leq m.$$

Furthermore denote by $g_\varepsilon(\langle \alpha_m \rangle)$ the steady state probability of exit from $\langle \alpha_m \rangle$, that is

$$g_\varepsilon(\langle \alpha_m \rangle) = \sum_{i^{(m)} \in X_m} \pi_\varepsilon(i^{(m)}) \sum_{j^{(m+1)} \in X_{m+1}} p_\varepsilon(i^{(m)}, j^{(m+1)}).$$

Denote by $\{\pi_0(i^{(0)}), i^{(0)} \in X_0\}$ the stationary distribution corresponding to P_0 and let

$$\bar{\pi}_0 = \{\pi_0(i^{(0)}), i^{(0)} \in X_0\}, \quad \bar{\pi}_\varepsilon^{(q)} = \{\pi_\varepsilon(i^{(q)}), i^{(q)} \in X_q\}$$

be row vectors. Finally, let the matrix

$$A^{(q)} = (\alpha^{(q)}(i^{(q)}, j^{(q+1)})), \quad i^{(q)} \in X_q, j^{(q+1)} \in X_{q+1}, \quad q = 0, \dots, m$$

defined by condition 2.

Conditions (1)-(4) enable us to compute the main terms of the asymptotic expression for $\bar{\pi}_\varepsilon^{(q)}$ and $g_\varepsilon(\langle \alpha_m \rangle)$. Namely, we obtain

$$\begin{aligned} \bar{\pi}_\varepsilon^{(q)} &= \varepsilon^q \bar{\pi}_0 A^{(0)} A^{(1)} \dots A^{(q-1)} + o(\varepsilon^q), \quad q = 1, \dots, m, \\ g_\varepsilon(\langle \alpha_m \rangle) &= \varepsilon^{(m+1)} \bar{\pi}_0 A^{(0)} A^{(1)} \dots A^{(m)} \underline{1} + o(\varepsilon^{m+1}), \end{aligned} \quad (1)$$

where $\underline{1} = (1, \dots, 1)^T$ is a column vector, [5, pp. 141-153].

Let $(\eta_\varepsilon(t), t \geq 0)$ be a Semi Markov Process (SMP) given by the embedded Markov chain $(X_\varepsilon(k), k \geq 0)$ satisfying conditions (1)-(4). Let the times $\tau_\varepsilon(j^{(s)}, k^{(z)})$ -transition times from state $j^{(s)}$ to state $k^{(z)}$ -fulfill the condition

$$E \exp \{i\theta \beta_\varepsilon \tau_\varepsilon(j^{(s)}, k^{(z)})\} = 1 + a_{jk}(s, z, \theta) \varepsilon^{m+1} + o(\varepsilon^{m+1}), \quad (i^2 = -1),$$

where β_ϵ is some normalizing factor. Denote by $\Omega_\epsilon(m)$ the instant at which the SMP reaches the $(m + 1)^{\text{th}}$ level for the first time, exit time from (α_m) , provided $\eta_\epsilon(0) \in (\alpha_m)$. Then we have:

THEOREM 1. [5] *If the above conditions are satisfied then*

$$\lim_{\epsilon \rightarrow 0} E \exp \{i\theta \beta_\epsilon \Omega_\epsilon(m)\} = (1 - A(\theta))^{-1}$$

where

$$A(\theta) = \frac{\sum_{j^{(0)}, k^{(0)} \in X_0} \pi_0(j^{(0)}) p_0(j^{(0)}, k^{(0)}) a_{jk}(0, 0, \theta)}{\bar{\pi}_0 A^{(0)} A^{(1)} \dots A^{(m)} \underline{1}}$$

COROLLARY 1. *In particular, if $a_{jk}(s, z, \theta) = i\theta m_{jk}(s, z)$ then the limit is an exponentially distributed random variable with mean*

$$\frac{\sum_{j^{(0)}, k^{(0)} \in X_0} \pi_0(j^{(0)}) p_0(j^{(0)}, k^{(0)}) m_{jk}(0, 0)}{\bar{\pi}_0 A^{(0)} A^{(1)} \dots A^{(m)} \underline{1}}$$

3. THE QUEUING MODEL

Consider a multiprocessor computer system consisting of N processors and a single bus. A processor that has a request to use a bus is said to be active, otherwise it is called inactive or idle. The service at the shared bus is carried out according to a work-conserving discipline. The processors are assumed to operate in a random environment governed by an ergodic Markov chain $(\xi_1(t), t \geq 0)$ with state space $(1, \dots, r_1)$ and with transition rate matrix $(a_{ij}, i, j = 1, \dots, r_1, a_{ii} = \sum_{j \neq i} a_{ij})$. Let us suppose that each processor can have at most one outstanding request at any time. Namely, at each processor a new request can be generated only after the bus usage of the previous request has been completed.

Whenever the environmental process is in state i the probability that an inactive processor becomes active, i.e., it generates a request, in the time interval $(t, t + h)$ is $\lambda(i, \epsilon)h + o(h)$. The buses are also supposed to operate in a random environment governed by an ergodic Markov chain $(\xi_2(t), t \geq 0)$ with state space $(1, \dots, r_2)$ and with transition rate matrix $(b_{kq}, k, q = 1, \dots, r_2, b_{kk} = \sum_{q \neq k} b_{kq})$. Whenever the environmental process is in state k the probability that a processor completes the bus usage, i.e., it is served by the bus, in time interval $(t, t + h)$ is $\mu(k)h + o(h)$.

All random variables involved here and the random environments are supposed to be independent of each other.

Let us consider the system under the assumption of "fast" arrivals, i.e., $\lambda(i, \epsilon) \rightarrow \infty$ as $\epsilon \rightarrow 0$. For simplicity let $\lambda(i, \epsilon) = \lambda(i)/\epsilon$. Denote by $Y_\epsilon(t)$ the number of inactive processors at time t , and let

$$\Omega_\epsilon(m) = \inf\{t : t > 0, Y_\epsilon(t) = m + 1 \mid Y_\epsilon(0) \leq m\},$$

that is, the instant at which the number of inactive processors reaches the $(m + 1)^{\text{th}}$ level for the first time, provided that at the beginning their number is not greater than $m, m = 1, \dots, N - 1$. In particular, if $m = N - 1$ then the bus becomes idle since there is no active processor, hence $\Omega_\epsilon(N - 1)$ can be referred to as the busy period length of the bus.

Denote by $(\pi_i^{(1)}, i = 1, \dots, r_1), (\pi_k^{(2)}, k = 1, \dots, r_2)$ the steady-state distribution of the governing Markov chains $(\xi_1(t), t \geq 0), (\xi_2(t), t \geq 0)$, respectively. Now we have:

THEOREM 2. *For the system in question under the above assumptions, independently of the initial state, the distribution of the normalized random variable $\epsilon^m \Omega_\epsilon(m)$ converges weakly to an exponentially distributed random variable with parameter*

$$\Lambda = \frac{1}{m!} \sum_{i=1}^{r_1} \sum_{k=1}^{r_2} \pi_i^{(1)} \pi_k^{(2)} \frac{\mu(k)^{m+1}}{\lambda(i)^m}.$$

PROOF. It is easy to see that the process

$$Z_\varepsilon(t) = (\xi_1(t), \xi_2(t), Y_\varepsilon(t))$$

is a three-dimensional Markov chain with state space

$$((i, k, s), \quad i = 1, \dots, r_1, \quad k = 1, \dots, r_2, \quad s = 0, \dots, N).$$

Furthermore, let

$$\langle \alpha_m \rangle = ((i, k, s), \quad i = 1, \dots, r_1, \quad k = 1, \dots, r_2, \quad s = 0, \dots, m).$$

Hence our aim is to determine the distribution of the first exit time of $Z_\varepsilon(t)$ from $\langle \alpha_m \rangle$, provided that $Z_\varepsilon(0) \in \langle \alpha_m \rangle$.

It can easily be verified that the transition probabilities in any time interval $(t, t+h)$ are the following:

$$(i, k, s) \xrightarrow{h} \begin{cases} (j, k, s) & a_{ij}h + o(h), & i \neq j, \\ (i, q, s) & b_{kq}h + o(h), & k \neq q, \\ (i, k, s+1) & \mu(k)h + o(h), & s = 0, \dots, N-1, \\ (i, k, s-1) & (s\lambda(i)/\varepsilon)h + o(h), & s = 1, \dots, N. \end{cases}$$

In addition, the sojourn time $\tau_\varepsilon(i, k, s)$ of $Z_\varepsilon(t)$ in state (i, k, s) is exponentially distributed with parameter

$$a_{ii} + b_{kk} + \frac{s\lambda(i)}{\varepsilon} + \mu(k), \quad s = 0, 1, \dots, N-1,$$

$$a_{ii} + b_{kk} + \frac{s\lambda(i)}{\varepsilon}, \quad s = 0.$$

Thus, the transition probabilities for the embedded Markov chain are

$$p_\varepsilon[(i, k, s), (j, k, s)] = \frac{a_{ij}}{a_{ii} + b_{kk} + s\lambda(i)/\varepsilon + \mu(k)}, \quad s = 0, 1, \dots, N-1,$$

$$p_\varepsilon[(i, k, N), (j, k, N)] = \frac{a_{ij}}{a_{ii} + b_{kk} + N\lambda(i)/\varepsilon}, \quad s = N,$$

$$p_\varepsilon[(i, k, s), (i, q, s)] = \frac{b_{kq}}{a_{ii} + b_{kk} + s\lambda(i)/\varepsilon + \mu(k)}, \quad s = 0, 1, \dots, N-1,$$

$$p_\varepsilon[(i, k, N), (i, q, N)] = \frac{b_{kq}}{a_{ii} + b_{kk} + N\lambda(i)/\varepsilon}, \quad s = N,$$

$$p_\varepsilon[(i, k, s), (i, k, s+1)] = \frac{\mu(k)}{a_{ii} + b_{kk} + s\lambda(i)/\varepsilon + \mu(k)}, \quad s = 0, \dots, N-1,$$

$$p_\varepsilon[(i, k, s), (i, k, s-1)] = \frac{s\lambda(i)/\varepsilon}{a_{ii} + b_{kk} + s\lambda(i)/\varepsilon + \mu(k)}, \quad s = 1, \dots, N-1,$$

$$p_\varepsilon[(i, k, N), (i, k, N-1)] = \frac{N\lambda(i)/\varepsilon}{a_{ii} + b_{kk} + N\lambda(i)/\varepsilon}, \quad s = N.$$

As $\varepsilon \rightarrow 0$ this implies

$$p_\varepsilon[(i, k, 0), (j, k, 0)] = \frac{a_{ij}}{a_{ii} + b_{kk} + \mu(k)},$$

$$p_\varepsilon[(i, k, 0), (i, q, 0)] = \frac{b_{kq}}{a_{ii} + b_{kk} + \mu(k)},$$

$$p_\varepsilon[(i, k, s), (j, k, s)] = o(1), \quad s = 1, \dots, N,$$

$$p_\varepsilon[(i, k, s), (i, q, s)] = o(1), \quad s = 1, \dots, N,$$

$$p_\varepsilon[(i, k, 0), (i, k, 1)] = \frac{\mu(k)}{a_{ii} + b_{kk} + \mu(k)},$$

$$p_\varepsilon[(i, k, s), (i, k, s+1)] = \frac{\mu(k)\varepsilon}{s\lambda(i)}(1 + o(1)), \quad s = 1, \dots, N-1,$$

$$p_\varepsilon[(i, k, s), (i, k, s-1)] = 1 - o(1), \quad s = 1, \dots, N.$$

This agrees with the conditions (1)–(4), but here the zero level is the set

$$((i, k, 0), (i, k, 1), \quad i = 1, \dots, r_1, \quad k = 1, \dots, r_2),$$

while the q^{th} level is the set

$$((i, k, q + 1), \quad i = 1, \dots, r_1, \quad k = 1, \dots, r_2).$$

Since the level 0 in the limit forms an essential class, the probabilities $\pi_o(i, k, 0)$, $\pi_o(i, k, 1)$, $i = 1, \dots, r_1$, $k = 1, \dots, r_2$ satisfy the following system of equations

$$\pi_o(j, q, 0) = \sum_{i \neq j} \frac{\pi_o(i, q, 0) a_{ij}}{(a_{ii} + b_{qq} + \mu(q))} \quad (2)$$

$$+ \sum_{k \neq q} \frac{\pi_o(j, k, 0) b_{kq}}{(a_{jj} + b_{kk} + \mu(k))} + \pi_o(j, q, 1)$$

$$\pi_o(j, q, 1) = \frac{\pi_o(j, q, 0) \mu(q)}{(a_{jj} + b_{qq} + \mu(q))}. \quad (3)$$

Since

$$\pi_j^{(1)} a_{jj} = \sum_{i \neq j} \pi_i^{(1)} a_{ij}, \quad \pi_q^{(2)} b_{qq} = \sum_{k \neq q} \pi_k^{(2)} b_{kq}, \quad (4)$$

it can easily be verified that the solution of (2), (3) subject to (4) is

$$\begin{aligned} \pi_o(i, k, 0) &= B \pi_i^{(1)} \pi_k^{(2)} (a_{ii} + b_{kk} + \mu(k)), \\ \pi_o(i, k, 1) &= B \pi_i^{(1)} \pi_k^{(2)} \mu(k) \end{aligned}$$

where B is the normalizing constant, i.e.,

$$\frac{1}{B} = \sum_{i=1}^{r_1} \sum_{k=1}^{r_2} \pi_i^{(1)} \pi_k^{(2)} (a_{ii} + b_{kk} + 2\mu(k)).$$

By using formula (1) it is easy to show that the probability of exit from $\langle \alpha_m \rangle$ is

$$\begin{aligned} g_\varepsilon(\langle \alpha_m \rangle) &= \varepsilon^m B \sum_{i=1}^{r_1} \sum_{k=1}^{r_2} \pi_i^{(1)} \pi_k^{(2)} \mu(k) \prod_{s=1}^m \frac{\mu(k)}{s\lambda(i)} (1 + o(1)) \\ &= \frac{\varepsilon^m B}{m!} \sum_{i=1}^{r_1} \sum_{k=1}^{r_2} \pi_i^{(1)} \pi_k^{(2)} \frac{\mu(k)^{m+1}}{\lambda(i)^m} (1 + o(1)). \end{aligned}$$

Taking into account the exponentiality of $\tau_\varepsilon(j, k, s)$ for fixed θ we have

$$\begin{aligned} E \exp \{i\varepsilon^m \theta \tau_\varepsilon(j, k, 0)\} &= 1 + \varepsilon^m \frac{i\theta}{a_{jj} + b_{kk} + \mu(k)} (1 + o(1)), \quad s = 0, \\ E \exp \{i\varepsilon^m \theta \tau_\varepsilon(j, k, s)\} &= 1 + o(\varepsilon^m), \quad s > 0. \end{aligned}$$

Notice that $\beta_\varepsilon = \varepsilon^m$ and therefore, from Corollary 1, we immediately get the statement that $\varepsilon^m \Omega_\varepsilon(m)$ converges weakly to an exponentially distributed random variable with parameter

$$\Lambda = \frac{1}{m!} \sum_{i=1}^{r_1} \sum_{k=1}^{r_2} \pi_i^{(1)} \pi_k^{(2)} \frac{\mu(k)^{m+1}}{\lambda(i)^m},$$

which completes the proof.

Consequently, the distribution of $\Omega_\epsilon(m)$ be approximated by

$$P(\Omega_\epsilon(m) > t) = P(\epsilon^m \Omega_\epsilon(m) > \epsilon^m t) \approx \exp(-\epsilon^m \wedge t),$$

i.e., $\Omega_\epsilon(m)$ is asymptotically an exponentially distributed random variable with parameter $\epsilon^m \wedge$.

In particular, for $m = N - 1$, we have

$$\epsilon^{N-1} \wedge = \epsilon^{N-1} \frac{1}{(N-1)!} \sum_{i=1}^{r_1} \sum_{k=1}^{r_2} \pi_i^{(1)} \pi_k^{(2)} \frac{\mu(k)^N}{\lambda(i)^{N-1}}.$$

In the case when there are no random environments, we get

$$\wedge^* = \epsilon^{N-1} \wedge = \frac{1}{(N-1)!} \frac{\mu^N}{(\lambda/\epsilon)^{N-1}}. \quad (5)$$

4. PERFORMANCE MEASURES

This section deals with the derivation of the main steady-state performance measures relating to the model.

4.1. Utilizations

The utilization U of the bus is defined as the fraction of time during which it is busy. The idle period of the bus starts when each processor is idle at the end of a service completion, and terminates when a processor generates a request. It is clear that the mean idle period length is

$$\sum_{i=1}^{r_1} \pi_i^{(1)} \frac{1}{N\lambda(i)/\epsilon}.$$

Hence, for U we obtain

$$U = \frac{1/(\epsilon^{N-1} \wedge)}{1/(\epsilon^{N-1} \wedge) + \sum_{i=1}^{r_1} \pi_i^{(1)} \frac{1}{N\lambda(i)/\epsilon}}. \quad (6)$$

The bus utilization U_p of processor p is defined as the fraction of time that processor p uses the bus. Since the processors are of the same kind for U_p we get

$$U_p = \frac{U}{N}, \quad p = 1, \dots, N. \quad (7)$$

4.2. Throughput

The throughput γ_p of processor p is defined as the mean number of requests of processor p served per unit time. It is well-known that $U_p = \gamma_p b_p$, where b_p is the mean bus usage (service) time of a request by processor p . In this case

$$U_p = \gamma_p \sum_{k=1}^{r_2} \pi_k^{(2)} \frac{1}{\mu(k)}, \quad \text{thus, } \gamma_p = \frac{U_p}{\sum_{k=1}^{r_2} \pi_k^{(2)} \frac{1}{\mu(k)}}.$$

4.3. Mean Delay, Mean Waiting Times

The mean delay T_p of processor p is the average time from the instant at which a request is generated at processor p to the instant at which the bus usage of that request has been completed. In other words, T_p is the mean duration of an active state at processor p . Since the state of processor p alternates between the active state of average duration T_p and the inactive state of mean duration

$$\sum_{i=1}^{r_1} \pi_i^{(1)} \frac{1}{\lambda(i)/\epsilon},$$

we have the following relationship

$$\gamma_p = \frac{1}{T_p + \sum_{i=1}^{r_1} \pi_i^{(1)} \frac{1}{\lambda(i)/\varepsilon}}.$$

Thus,

$$T_p = \frac{1}{\gamma_p} - \sum_{i=1}^{r_1} \pi_i^{(1)} \frac{1}{\lambda(i)/\varepsilon}.$$

Furthermore, for the mean waiting time W_p of processor p , we obtain

$$W_p = T_p - \sum_{k=1}^{r_2} \pi_k^{(2)} \frac{1}{\mu(k)}.$$

4.4. Average Number of Requests Served During a Busy Period

A pair of an idle period followed by a busy period is called a cycle, whose mean length is denoted by C . Clearly,

$$C = \frac{1}{\varepsilon^{N-1}\Lambda} + \sum_{i=1}^{r_1} \pi_i^{(1)} \frac{1}{N\lambda(i)/\varepsilon}.$$

Denote by N_p the mean number of requests of processor p served during a cycle. The throughput γ_p of processor p is then given by $\gamma_p = N_p/C$, which yields that the total number of requests served during a busy period is

$$\sum_{p=1}^N N_p = \sum_{p=1}^N \gamma_p C = \frac{1/(\varepsilon^{N-1}\Lambda)}{\sum_{k=1}^{r_2} \pi_k^{(2)} \frac{1}{\mu(k)}},$$

since all measures mentioned above are independent of p .

4.5. Mean Number of Active Processors

Let us denote by $U^{(p)}$ the steady-state probability that processor p is idle. Clearly, we have

$$U^{(p)} = \frac{\sum_{i=1}^{r_1} \pi_i^{(1)} \frac{1}{\lambda(i)/\varepsilon}}{T_p + \sum_{i=1}^{r_1} \pi_i^{(1)} \frac{1}{\lambda(i)/\varepsilon}} = \gamma_p \sum_{i=1}^{r_1} \pi_i^{(1)} \frac{1}{\lambda(i)/\varepsilon}.$$

Hence, the mean number of active processors is

$$\sum_{p=1}^N 1 - U^{(p)} = N - \frac{U}{\sum_{k=1}^{r_2} \pi_k^{(2)} \frac{1}{\mu(k)}} \sum_{i=1}^{r_1} \pi_i^{(1)} \frac{1}{\lambda(i)/\varepsilon},$$

which tends to N as $\lambda(i)/\varepsilon \rightarrow \infty$.

5. NUMERICAL RESULTS

This section presents a number of validation experiments (Tables 1-8) examining the credibility of the proposed approximation against exact results for the performance measure of processor utilization at equilibrium. Note that an exact formula for the utilization is known only when the system is not effected by random environment and it is given (via Palm-formula) by

$$U_p^* = \frac{1}{N} \frac{\sum_{k=1}^N \binom{N}{k} k! \rho^k}{1 + \sum_{k=1}^N \binom{N}{k} k! \rho^k}, \quad \text{where } \rho = \frac{\lambda/\varepsilon}{\mu}$$

Table 1.

N=3		
ρ	U_p^*	U_p
1	0.3125	0.285714286
2	0.329113924	0.326530612
2^2	0.332657201	0.332467532
2^3	0.333237575	0.333224862
2^4	0.333320592	0.333319771
2^5	0.333333169	0.333331638
2^6	0.333333125	0.333333121
2^7	0.333333307	0.333333307
2^8	0.333333333	0.333333333

Table 2.

N=4		
ρ	U_p^*	U_p
1	0.246153846	0.24
2	0.249605055	0.249350649
2^2	0.249968310	0.249959317
2^3	0.249997756	0.249997457
2^4	0.249999999	0.249999999
2^5	0.25	0.25

Table 3.

N=5		
ρ	U_p^*	U_p
1	0.199386503	0.198347107
2	0.199968409	0.199947930
2^2	0.199998732	0.199998372
2^3	0.199999955	0.199999949
2^4	0.199999998	0.199999998
2^5	0.2	0.2

Table 4.

N=6		
ρ	U_p^*	U_p
1	0.166581502	0.166435506
2	0.166664473	0.166666305
2^2	0.166666623	0.166666661
2^3	0.166666666	0.166666666

Table 5.

N=7		
ρ	U_p^*	U_p
1	0.142846715	0.142828804
2	0.142857009	0.142856921
2^2	0.142857142	0.142857141
2^3	0.142857143	0.142857143

Table 6.

N=8		
ρ	U_p^*	U_p
1	0.124998860	0.1249969
2	0.124999993	0.124999988
2^2	0.125	0.125

Table 7.

N=9		
ρ	U_p^*	U_p
1	0.111110998	0.111110805
2	0.111111111	0.111111111

Table 8.

N=10		
ρ	U_p^*	U_p
1	0.099999999	0.099999999
2	0.1	0.1

In this case relations (5-7) reduce to the following approximation:

$$U_p = \frac{1}{N} \frac{N!}{N! + \left(\frac{\mu}{\lambda/\varepsilon}\right)^N}.$$

It can be observed from the results shown above that the approximate values for $\{U_p\}$ are very much comparable in accuracy to those provided by the exact results for $\{U_p^*\}$. However, the computational complexity, due to the proposed approximation, has been considerably reduced. As λ/ε becomes greater than μ , the $\{U_p\}$ approximations, as expected, approach the exact values of $\{U_p^*\}$. Clearly, the greater the number of processors the less number of steps are needed to reach the exact results.

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