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Theory and Methodology

Asymptotic analysis of some complex renewable systems operating in random environments

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Abstract: The present paper is concerned with an asymptotic analysis of some complex renewable systems operating in random environments. Assuming 'fast' repair, it is shown that the time to the first system failure, converges in distribution under appropriate norming to an exponentially distributed random variable.

Keywords: Operating time, repair time, fast repair, random environments, system failure, weak convergence

1. Introduction

The ultimate goal of reliability theory is to give a numerical estimate of reliability indices. It is well known that for more or less complicated cases an exact reliability evaluation is practically impossible. This stimulates interest in approximate methods in reliability calculations. In many models of great practical interest 'small parameters' are usually present, e.g. the element failure rates are much smaller than their repair rates. (This is termed in reliability theory as 'fast' repair.) The measure of greatest interest is the probability of failure-free operation of the system during a given time t. Different methods and approaches have been developed in order to that the involved models should be mathematically tractable. For good reviews and materials the interested reader is referred to, among others, Birolini [3], Burtin and Pittel [4], Franken et al. [5], Gertsbakh [6], Gnedenko and Solovjev [7], Gnedenko [8], Kozlov and Ushakov [11], Kovalenko [12], and Ushakov [13].

It is also well known that a great majority of problems can be treated by the help of Semi-Markov Processes (SMP). Since the failure-free operation of the system corresponds to sojourn time problems for SMP we can use the results obtained for SMP. It is easy to see, that in the case of 'fast repair' the exit from a given subset of the state space of the underlying SMP is a 'rare' event, that is, it occurs with a small probability. Thus, it is natural to investigate the asymptotic behavior of sojourn time in a given subset, provided that the probability of exit from it tends to zero, see e.g. Anisimov [1], Anisimov et al. [2], Keilson [9], and Koroljuk and Turbin [10].

The purpose of the present paper is two-fold. On the one hand, without proofs we give a brief survey of preliminary results due to Anisimov, see [1,2,15]. On the other hand, we deal with an asymptotic analysis

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of some renewable systems operating in random environments. Assuming 'fast' repair, it is shown that the time to the first system failure converges in distribution under appropriate norming to an exponentially distributed random variable. The main contribution of the paper is the following. The failure and repair intensities of the elements depend on the state of the system operating in random environments. As a result of this assumption, the corresponding subset of the limiting Markov process—constructed for this problem—is not a simple essential class of states. Hence, the 'classical' methods cannot be applied. Using the results of [1,2], the main term of the steady-state probability of exit from the given subset can be calculated. Then, by the help of the Theorems cited in Section 2 the asymptotic exponentiality is proved.

2. Preliminary results

Let $(\xi(t),\ t\geqslant 0)$ be a semi-Markov process with state space $\{0, 1,\dots, r\}$ given by the embedded Markov chain $(X_n,\ n\geqslant 0)$ and by the transition matrix $\| p(i,\ j)\|,\ i,\ j=0,\dots,r$. Furthermore, let $\tau(i,\ j)$ be mutually independent random variables denoting the time spent in state i, given that the next state is j, $i,\ j=0,\dots,r$. Let $\Omega(k)$ denote the sojourn time of $\xi(t)$ in subset $\{1,\dots,r\}$ started in state k, that is

$$\Omega(k)=\inf\{t\colon\ t>0,\ \xi(t)=0\mid\xi(0)=k,\ k\neq 0\}.$$

Suppose that $X_n=X_\varepsilon(n)$, $p(k,\ j)=p_\varepsilon(k,\ j)$ and $\tau(k,\ j)=\tau_\varepsilon(k,\ j)$, that is $(\xi(t),\ t\geqslant 0)$ depends on some small parameter ε, such that $p_\varepsilon(k, 0)\to 0$ as $\varepsilon\to 0$. Therefore, it is natural to investigate $\Omega_\varepsilon(k)$ as $\varepsilon\to 0$. In order to state the relevant theorems, let us introduce some important notions. Let $\langle\alpha\rangle$ be a subset from $\{1,\dots,r\}$, and let

$$v_\varepsilon(i,\langle\alpha\rangle)=\min\{k\colon\ k>0,\ X_\varepsilon(k)\notin\langle\alpha\rangle\mid X_\varepsilon(0)\in\langle\alpha\rangle\},$$
$$q_\varepsilon(i,\ j,\ \langle\alpha\rangle)=P\{X_\varepsilon(l)=j \text{ for at least one } l,\ l<v_\varepsilon(i,\langle\alpha\rangle)\mid X_\varepsilon(0)=i\},$$

that is, $q_\varepsilon(i,\ j,\ \langle\alpha\rangle)$ is the probability of a visit to j up to the time when the chain exists from $\langle\alpha\rangle$, given that the initial state was $i,\ j\in\langle\alpha\rangle$. $\langle\alpha\rangle$ is called an s-set (communicating set) if for any $i,\ j\in\langle\alpha\rangle$, $q_\varepsilon(i,\ j,\ \langle\alpha\rangle)\to 0$ as $\varepsilon\to 0$.

Practically, it means that initiated from any state the chain visits each state asymptotically infinitely many times before leaving $\langle\alpha\rangle$. (The simplest example for an s-set is a set which in the limit forms a simple essential class.)

Let assume that the subject $\{1,\dots,r\}$ forms an s-set and let

$$g_\varepsilon=\sum_{k=1}^{r}\Pi_\varepsilon(k)p_\varepsilon(k, 0),$$

where $\Pi_\varepsilon(k)$, $k=1,\dots,r$, is the stationary distribution for the chain with transition matrix

$$\| p_\varepsilon(i,\ j)/(1-p_\varepsilon(i, 0))\|,\quad i,\ j=1,\dots,r.$$

Furthermore, suppose that

$$\Pi_\varepsilon(k)p_\varepsilon(k, 0)/g_\varepsilon\to b_k,\quad k=1,\dots,r,$$

and there exists a normalizing factor β_ε such that

(a) $\mathrm{E}\exp\{iu\beta_\varepsilon\tau(k,\ j)\}=1+g_\varepsilon a_{kj}(u)+o(g_\varepsilon),\quad k,\ j=1,\dots,r,$

(b) $\mathrm{E}\exp\{iu\beta_\varepsilon\tau(k, 0)\}=\rho_k(u),\quad k=1,\dots,r.$

Theorem 1 (Anisimov [1,2]). *If the above conditions are satisfied, then independently of the initial state j, $j=1,\dots,r$, the distribution of $\beta_\varepsilon\Omega_\varepsilon(j)$ converges weakly to a distribution with characteristic function*

$$\left(\sum_{k=1}^{r}b_k\rho_k(u)\right)\Big/\left(1-\sum_{k,j=1}^{r}\Pi_0(k)p_0(k,\ j)a_{kj}(u)\right),$$

where

$$\Pi_0(k)=\lim_{\varepsilon\to 0}\Pi_\varepsilon(k),\qquad p_0(k,\,j)=\lim_{\varepsilon\to 0}p_\varepsilon(k,\,j),\quad k,\,j=1,\dots,r.$$

The most crucial part of applying Theorem 1 to particular situations is finding the normalizing factor β_ε.

In the following an example is given on which our further considerations are based.

Example (Anisimov et al. [2, pp. 151]). Let $X_\varepsilon(k)$, $k\geqslant 0$ be a Markov chain with state space

$$E=\{(i,\,q),\ i=1,\dots,r,\ q=0,\dots,m+1\},$$

defined by the transition matrix $\|\,p_\varepsilon[(i,\,q),\,(j,\,z)]\,\|$ satisfying the following conditions:

(1) $p_\varepsilon[(i,0),(j,0)]\to p_{ij}$, $i,\ j=1,\dots,r$, and the matrix $\|\,p_{ij}\,\|$, $i,\ j=1,\dots,r$, is irreducible,

(2) $p_\varepsilon[(i,\,q),(j,\,q+1)]=\varepsilon\alpha_{ij}^{(q)}+o(\varepsilon)$, $i,\ j=1,\dots,r$, $q=0,\dots,m$,

(3) $p_\varepsilon[(i,\,q),(j,\,q)]\to 0$, $i,\ j=1,\dots,r$, $q\geqslant 1$,

(4) $p_\varepsilon[(i,\,q),(j,\,z)]\equiv 0$, $i,\ j=1,\dots,r$, $|z-q|\geqslant 2$.

In the sequel the set of states $\{(i,\,q),\ i=1,\dots,r\}$ is called the q-th level of the chain, $q=0,\dots,m+1$. Let us single out the subset of states

$$\langle\alpha\rangle=\{(i,\,q),\ i=1,\dots,r,\ q=0,\dots,m\}.$$

Denote by $\Pi_\varepsilon(i,\,q)$ the stationary distribution of $X_\varepsilon(k)$ and by $g_\varepsilon(\langle\alpha\rangle)$ the steady-state probability of exit from $\langle\alpha\rangle$, that is

$$g_\varepsilon(\langle\alpha\rangle)=\sum_{i=1}^{r}\Pi_\varepsilon(i,\,m)\sum_{j=1}^{r}p_\varepsilon[(i,\,m),\,(j,\,m+1)].$$

Let

$$P=\|\,p_{ij}\,\|,\quad i,\ j=1,\dots,r,\qquad A^{(q)}=\left\|\alpha_{ij}^{(q)}\right\|,\quad i,\ j=1,\dots,r,\quad q=0,\dots,m,$$

$\{\Pi_k,\ k=1,\dots,r\}$ be the stationary distribution for the chain with matrix P, and

$$\overline{\Pi}=(\Pi_1,\dots,\Pi_r),\qquad \overline{\Pi}_\varepsilon^{(q)}=\left(\Pi_\varepsilon^{(q)}(1,\,q),\dots,\Pi_\varepsilon^{(q)}(r,\,q)\right)$$

be row vectors. Conditions (1)–(4) enable us to compute the main terms of the asymptotic expression for $\overline{\Pi}_\varepsilon^{(q)}$ and $g_\varepsilon(\langle\alpha\rangle)$, namely, we obtain

$$\begin{aligned}\overline{\Pi}_\varepsilon^{(q)}&=\varepsilon^q\overline{\Pi}A^{(0)}\cdots A^{(q-1)}+o(\varepsilon^q),\quad q\geqslant 1,\\ g_\varepsilon(\langle\alpha\rangle)&=\varepsilon^{m+1}\overline{\Pi}A^{(0)}\cdots A^{(m)}\mathbf{1}+o(\varepsilon^{m+1}),\end{aligned}\tag{1}$$

where $\mathbf{1}=(1,\dots,1)^{\mathrm{T}}$.

Now, making use of Theorem 1 and formula (1) we get the following asymptotic result.

Let $(\xi_\varepsilon(t),\ t\geqslant 0)$ be a SMP given by the embedded Markov chain $(X_\varepsilon(k),\ k\geqslant 0)$ satisfying conditions (1)–(4). Let the times $\tau_\varepsilon[(l,\,s),(j,\,z)]$, the transition times from state $(l,\,s)$ to state $(j,\,z)$, fulfill the condition

$$\mathrm{E}\exp\{\mathrm{i}\theta\beta_\varepsilon\tau_\varepsilon[(l,\,s),\,(j,\,z)]\}=1+a_{lj}(s,\,z,\,\theta)\varepsilon^{m+1}+o(\varepsilon^{m+1}),$$

where β_ε is some normalizing factor.

Denote by $\Omega_\varepsilon^{(m)}(j,\,s)$ the instant at which the SMP reaches the $(m+1)$-th level for the first time, provided $\xi_\varepsilon(0)=(j,\,s)$, $s\leqslant m$.

Corollary 1. *If the above conditions are satisfied, then*

$$\lim_{\varepsilon \to 0} \mathrm{E} \exp\left\{ \mathrm{i}\theta\beta_\varepsilon \Omega_\varepsilon(j, s) \right\} = (1 - A(\theta))^{-1},$$

where

$$A(\theta) = \left(\sum_{k,j=1}^{r} \Pi_k p_{kj} a_{kj}(0, 0, \theta) \right) / (\overline{\Pi} A^{(0)} \cdots A^{(m)} \mathbf{1}).$$

In particular, if $a_{ij}(s, z, \theta) = \mathrm{i}\theta m_{ij}(s, z)$ then the limit is an exponentially distributed random variable with parameter

$$(\overline{\Pi} A^{(0)} \cdots A^{(m)} \mathbf{1}) / \left(\sum_{k,j=1}^{r} \Pi_k p_{kj} m_{kj}(0, 0) \right).$$

3. The mathematical model

Let us consider a renewable system consisting of n elements and one repairman. The expected life time of the l-th element is assumed to be an exponentially distributed random variable with failure rate λ_l. When the elements fail, they enter a repair facility and will be immediately served, unless the repairman is busy; otherwise they will wait in a queue in the order of their breakdowns. The required repair time of the l-th element is supposed to be an exponentially distributed random variable with service rate $\mu_l(\varepsilon)$, $l = 1, \ldots, n$.

Furthermore, we assume that the system is evolving in two random environments governed by irreducible, aperiodic Markov chains $(X_1(t), t \geqslant 0)$, $(X_2(t), t \geqslant 0)$ with state spaces $\{1, \ldots, r_1\}$, $\{1, \ldots, r_2\}$ and with transition density matrices

$$\left\{ a_{i_1 j_1},\ i_1,\ j_1 = 1, \ldots, r_1,\ a_{i_1 i_1} = \sum_{k \neq i_1} a_{i_1 k} \right\},$$

$$\left\{ b_{i_2 j_2},\ i_2,\ j_2 = 1, \ldots, r_2,\ b_{i_2 i_2} = \sum_{k \neq i_2} b_{i_2 k} \right\},$$

respectively. Whenever $X_1(t) = i_1$ and at time t there are s, $s = 0, \ldots, n-1$, elements with indices $(k_1, \ldots, k_s)$ at the service facility, the probability of the l-th element failure in the interval $(t, t+h)$ is $\lambda_l c(i_1\colon k_1, \ldots, k_s)h + o(h)$, $l \neq k_1, \ldots, k_s$. Similarly, whenever $X_2(t) = i_2$ and at time t there are s, $s = 1, \ldots, n$, elements with indices $(k_1, \ldots, k_s)$ at the service facility, the probability of the k_1-th element repair in the interval $(t, t+h)$ is $\mu_{k_1}(\varepsilon) d(i_2\colon k_1, \ldots, k_s)h + o(h)$. The environmental processes, operating and repair times are supposed to be independent of each other.

Let us consider the system under the assumption of 'fast' service, that is $\mu_l(\varepsilon) \to \infty$ as $\varepsilon \to 0$. For simplicity let $\mu_l(\varepsilon) = \mu_l/\varepsilon$, $l = 1, \ldots, n$. The system is said to have failed if the number of failed elements is $m = 1$, $1 < m < n$. Our goal is to determine the distribution of the failure-free operation time of the system.

Therefore, construct the following multi-dimensional Markov process:

$$Z_\varepsilon(t) = \left\{ X_1(t),\ X_2(t)\colon\ y_\varepsilon(t);\ \gamma_1(t), \ldots, \gamma_{y_\varepsilon(t)}(t) \right\}$$

with state space

$$E = \left\{ (i_1,\ i_2;\ s\colon\ k_1, \ldots, k_s);\ i_1 = 1, \ldots, r_1,\ i_2 = 1, \ldots, r_2,\ s = 0, \ldots, n, \right.$$
$$\left. (k_1, \ldots, k_s) \in V_n^s,\ k_0 = 0 \right\},$$

where $X_1(t)$, $X_2(t)$ are governing Markov chains; $y_\varepsilon(t)$ is the number of failed elements at time t; $\gamma_1(t),\dots,\gamma_{y_\varepsilon(t)}(t)$ are indices of the failed elements at time t in the order of their breakdowns; V_n^s is set of all variations of order s of integers $1,\dots,n$, and $k_0=0$ means the repairman is idle.

Let us single out the subset of states

$$\langle\alpha\rangle=\left\{\left(i_1,\, i_2;\, s\colon k_1,\dots,k_q\right),\ i_1=1,\dots,r_1,\ i_2=1,\dots,r_2,\ q=0,\dots,m,\ \left(k_1,\dots,k_q\right)\in V_n^q\right\}.$$

Let

$$\Omega_\varepsilon(m)=\inf\{t\colon t>0,\ y_\varepsilon(t)=m+1\mid y_\varepsilon(0)\leqslant m\},$$

that is, the instant at which the system break downs for the first time. Hence, the problem is to determine the distribution of the first exit time of $Z_\varepsilon(t)$ from $\langle\alpha\rangle$. Before going into details let us introduce some further notation. Let

$$\lambda_l c(i_1\colon k_1,\dots,k_s)=\lambda_l(i_1\colon k_1,\dots,k_s),\qquad \mu_l d(i_2\colon k_1,\dots,k_s)=\mu_l(i_2\colon k_1,\dots,k_s),$$
$$a_{i_1 i_1}+b_{i_2 i_2}+\sum_{l\neq k_1,\dots,k_s}\lambda_l(i_1\colon k_1,\dots,k_s)+\mu_{k_1}(i_2\colon k_1,\dots,k_s)/\varepsilon=\operatorname{Nom}(i_1,\, i_2\colon k_1,\dots,k_s).$$

It is easy to see that the sojourn time of $Z_\varepsilon(t)$ in state $(i_1,\, i_2;\, s\colon k_1,\dots,k_s)$ is exponentially distributed with parameter $\operatorname{Nom}(i_1,\, i_2\colon k_1,\dots,k_s)$. Furthermore, it can readily be verified that the transition probabilities for the embedded Markov chain are

$$p_\varepsilon\left[(i_1,\, i_2;\, s\colon k_1,\dots,k_s),\, (j_1\, i_2;\, s\colon k_1,\dots,k_s)\right]=\mathrm{o}(1),\quad s\geqslant 1,$$
$$p_\varepsilon\left[(i_1,\, i_2;\, s\colon k_1,\dots,k_s),\, (i_1,\, j_2;\, s\colon k_1,\dots,k_s)\right]=\mathrm{o}(1),\quad s\geqslant 1,$$
$$p_\varepsilon\left[(i_1,\, i_2;\, s\colon k_1,\dots,k_s),\, (i_1,\, i_2;\, s+1\colon k_1,\dots,k_{s+1})\right]$$
$$=\left(\lambda_{k_{s+1}}(i_1\colon k_1,\dots,k_s)\varepsilon/\mu_{k_1}(i_2\colon k_1,\dots,k_s)\right)(1+\mathrm{o}(1)),$$
$$p_\varepsilon\left[(i_1,\, i_2;\, s\colon k_1,\dots,k_s),\, (i_1,\, i_2;\, s-1\colon k_2,\dots,k_s)\right]\to 1,\quad 1\leqslant s\leqslant n,$$
$$p_\varepsilon\left[(i_1,\, i_2;\, 0\colon 0),\, (j_1,\, i_2;\, 0\colon 0)\right]=a_{i_1 j_1}/\operatorname{Nom}(i_1,\, i_2\colon 0),$$
$$p_\varepsilon\left[(i_1,\, i_2;\, 0\colon 0),\, (i_1,\, j_2;\, 0\colon 0)\right]=b_{i_2 j_2}/\operatorname{Nom}(i_1,\, i_2\colon 0),$$
$$p_\varepsilon\left[(i_1,\, i_2;\, 0\colon 0),\, (i_1,\, i_2;\, 1\colon k)\right]=\lambda_k(i_1\colon 0)/\operatorname{Nom}(i_1,\, i_2\colon 0).$$

This agrees with the conditions (1)–(4) of the Example, but there the zero level is the set

$$\left\{(i_1,\, i_2;\, 0\colon 0),\, (i_1,\, i_2;\, 1\colon k),\ i_1=1,\dots,r_1,\ i_2=1,\dots,r_2,\ k=1,\dots,n\right\},$$

while the q-th level is

$$\left\{\left(i_1,\, i_2;\, q+1\colon k_1,\dots,k_{q+1}\right),\ i_1=1,\dots,r_1,\ i_2=1,\dots,r_2,\ \left(k_1,\dots,k_{q+1}\right)\in V_n^{q+1}\right\}.$$

Since the level 0 in the limit forms an essential class, the probabilities $\Pi_0(i_1,\, i_2;\, 0\colon 0)$, $\Pi_0(i_1,\, i_2;\, 1\colon k)$ satisfy the following system of equations:

$$\Pi_0(i_1,\, i_2;\, 0\colon 0)=\sum_{j_1\neq i_1}\Pi_0(j_1,\, i_2;\, 0\colon 0)\frac{a_{j_1 i_1}}{\operatorname{Nom}(j_1,\, i_2\colon 0)}$$
$$+\sum_{j_2\neq i_2}\Pi_0(i_1,\, j_2;\, 0\colon 0)\frac{b_{j_2 i_2}}{\operatorname{Nom}(i_1,\, j_2\colon 0)}+\sum_{k=1}^{n}\Pi_0(i_1,\, i_2;\, 1\colon k),\tag{2}$$

$$\Pi_0(i_1,\, i_2;\, 1\colon k)=\Pi_0(i_1,\, i_2;\, 0\colon 0)\lambda_k(i_1\colon 0)/\operatorname{Nom}(i_1,\, i_2\colon 0).\tag{3}$$

Denote by $(\Pi^{(1)}_{i_1},\ i_1=1,\dots,r_1)$, $(\Pi^{(2)}_{i_2},\ i_2=1,\dots,r_2)$ the stationary distribution of the governing Markov chains $(X_1(t),\ t\geqslant 0)$, $(X_2(t),\ t\geqslant 0)$, respectively. Clearly,

$$\Pi^{(1)}_{i_1}a_{i_1i_1}=\sum_{j_1\neq i_1}\Pi^{(1)}_{j_1}a_{j_1i_1},\quad i_1=1,\dots,r_1,\tag{4}$$

$$\Pi^{(2)}_{i_2}b_{i_2i_2}=\sum_{j_2\neq i_2}\Pi^{(2)}_{j_2}b_{j_2i_2},\quad i_2=1,\dots,r_2.\tag{5}$$

It is not difficult to verify that the solution of (2), (3) subject to (4), (5) is

$$\Pi_0(i_1,\ i_2;\ 0\colon 0)=B\Pi^{(1)}_{i_1}\Pi^{(2)}_{i_2}\ \mathrm{Nom}(i_1,\ i_2\colon 0),\qquad \Pi_0(i_1,\ i_2;\ 1\colon k)=B\Pi^{(1)}_{i_1}\Pi^{(2)}_{i_2}\lambda_k(i_1\colon 0),\tag{6}$$

where

$$B=\left[\sum_{i_1=1}^{r_1}\sum_{i_2=1}^{r_2}\Pi^{(1)}_{i_1}\Pi^{(2)}_{i_2}\left(2\sum_{l=1}^{n}\lambda_l(i_1\colon 0)+a_{i_1i_1}+b_{i_2i_2}\right)\right]^{-1}.$$

Hence, denoting by $\Pi_\varepsilon(i_1,\ i_2;\ q\colon k_1,\dots,k_q)$ the steady-state probability of state $(i_1,\ i_2;\ q\colon k_1,\dots,k_q)$, according to (1) we obtain

$$\begin{aligned}&\Pi_\varepsilon(i_1,\ i_2;\ q\colon k_1,\dots,k_q)\\&\quad=\varepsilon^{q-1}B\Pi^{(1)}_{i_1}\Pi^{(2)}_{i_2}\frac{\lambda_{k_1}(i_1\colon 0)\lambda_{k_2}(i_1\colon k_1)\cdots\lambda_{k_q}(i_1\colon k_1,\dots,k_{q-1})}{\mu_{k_1}(i_2\colon k_1)\mu_{k_1}(i_2\colon k_1,\ k_2)\cdots\mu_{k_1}(i_2\colon k_1,\dots,k_{q-1})}(1+\mathrm{o}(1))\end{aligned}\tag{7}$$

and

$$g_\varepsilon(\langle\alpha\rangle)=\sum_{i_1=1}^{r_1}\sum_{i_2=1}^{r_2}\sum_{(k_1,\dots,k_{m+1})\in V_n^{m+1}}\Pi_\varepsilon(i_1,\ i_2;\ 1\colon k_1,\dots,k_{m+1}).\tag{8}$$

Taking into consideration the exponentiality of sojourn times of $Z_\varepsilon(t)$ in state $(i_1,\ i_2;\ s\colon k_1,\dots,k_s)$ for fixed u we obtain

$$\mathrm{E}\exp\{\mathrm{i}\varepsilon^m u\tau_\varepsilon(i_1,\ i_2;\ 0\colon 0)\}=1+(\varepsilon^m\mathrm{i}u/\mathrm{Nom}(i_1,\ i_2;\ 0\colon 0))(1+\mathrm{o}(1)),$$

$$\mathrm{E}\exp\{\mathrm{i}\varepsilon^m u\tau_\varepsilon(i_1,\ i_2;\ s\colon k_1,\dots,k_s)\}=1+\mathrm{o}(\varepsilon^m),\quad s>0,\quad (k_1,\dots,k_s)\in V_n^s.$$

Notice that $\beta_\varepsilon=\varepsilon^m$. Therefore, by the help of Corollary 1 we immediately get the following theorem.

Theorem 2. *For the system in question, under the above assumptions, independently of the initial state, the distribution of the normalized random variable $\varepsilon^m\Omega_\varepsilon(m)$ converges weakly to an exponentially distributed random variable with parameter*

$$\Lambda=\sum_{i_1=1}^{r_1}\sum_{i_2=1}^{r_2}\sum_{(k_1,\dots,k_{m+1})}\Pi^{(1)}_{i_1}\Pi^{(2)}_{i_2}\frac{\lambda_{k_1}(i_1\colon 0)\lambda_{k_2}(i_1\colon k_1)\cdots\lambda_{k_{m+1}}(i_1\colon k_1,\dots,k_m)}{\mu_{k_1}(i_2\colon k_1)\mu_{k_1}(i_2\colon k_1,\ k_2)\cdots\mu_{k_1}(i_2\colon k_1,\dots,k_m)}.\tag{9}$$

In particular, if the rates depend only on the number of failed elements and the states of the corresponding environmental processes, then (9) assumes the form

$$\Lambda=\sum_{i_1=1}^{r_1}\sum_{i_2=1}^{r_1}\sum_{(k_1,\dots,k_{m+1})}\Pi^{(1)}_{i_1}\Pi^{(2)}_{i_2}\frac{\lambda_{k_1}(i_1\colon 0)\lambda_{k_2}(i_1\colon 1)\cdots\lambda_{k_{m+1}}(i_1\colon m)}{\mu_{k_1}(i_2\colon 1)\mu_{k_1}(i_2\colon 2)\cdots\mu_{k_1}(i_2\colon m)}.$$

In addition, without environmental processes for Λ we have

$$\Lambda=\sum_{(k_1,\dots,k_{m+1})}\frac{\lambda_{k_1}(0)\lambda_{k_2}(1)\cdots\lambda_{k_{m+1}}(m)}{\mu_{k_1}(1)\mu_{k_1}(2)\cdots\mu_{k_1}(m)}.$$

In the case of homogeneous elements for Λ we obtain

$$\Lambda = \binom{n}{m+1}(m+1)!\lambda(0)\prod_{l=1}^{m}\frac{\lambda(l)}{\mu(l)}.$$

Finally, for the simplest case we get

$$\Lambda = n(n-1)\cdots(n-m)\lambda\left(\frac{\lambda}{\mu}\right)^{m},$$

which agrees with [14].

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