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An efficient method to solve a two-server heterogeneous retrial queue with threshold policy

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Efrosinin and Sztrik [**1**] considered a two-server heterogeneous retrial queue with threshold policy. They model the system as a quasi-birth-and-death (QBD) process with threshold dependent block-tridiagonal infinitesimal matrix and apply the general theory of matrix-geometric solutions. Thus, the computation of the rate matrix R (the minimal non-negative solution to the matrix equation) is based on the iteration algorithm. However, R can be expressed in a closed formula.

1.1. Level-dependent Band-QBD Solution. From the form of Λ (see page 214 of [**1**]) we observe that has two QBD-bands (see [**2**]):

- In band 1, we have the balance equations

$$\pi_{i-1}A_{01} + \pi_i A_{10} + \pi_{i+1}A_{21} = 0, \;\; 1 \le i \le q_2 - 2. \tag{1}$$

- the balance equations of band 2 is expressed as follows.

$$\pi_{i-1}A_{02} + \pi_i A_{12} + \pi_{i+1}A_{22} = 0, \;\; i \ge q_2 + 1. \tag{2}$$

Note that if the size of the orbit is finite, then the range of index i in equation (2) is upper bounded by $N-1$. That is, $q_2 + 1 \le i \le N-1$.

1.2. Infinite Orbit Size. The boundary balance equations are

$$\pi_0 A_{00} + \pi_1 A_{21} = 0, \tag{3}$$

$$\pi_{q_2-2}A_{01} + \pi_{q_2-1}A_{11} + \pi_{q_2}A_{22} = 0. \tag{4}$$

MGM (Matrix Geometric Method):

From eq. (1), we obtain (see [**3, 4**])

$$\pi_i = \pi_0 R_1^i + \pi_{q_2-1}R_2^{q_2-1-i}, \;\; \forall i = 0, \ldots, q_2 - 1, \tag{5}$$

where R_1 and R_2 are the minimal non-negative solution of the quadratic matrix equations

$$A_{01} + R_1 A_{10} + R_1^2 A_{21} = 0, \tag{6}$$

$$A_{21} + R_2 A_{10} + R_2^2 A_{01} = 0, \tag{7}$$

respectively.

The matrix geometric solution for π_i, $i \ge q_2$ is given by

$$\pi_i = \pi_{q2} R^{i-q2}, \;\; i \ge q_2, \tag{8}$$

where R is the minimal non-negative solution of the quadratic matrix equation

$$A_{02} + RA_{12} + R^2 A_{22} = 0. \tag{9}$$

SE (Spectral Expansion):

Following [**2**], we obtain the expression for π_i, $0 \le i \le q_2 - 1$, from equation (1)

$$\pi_i = \sum_{k=1}^{4} a_{1,k}\boldsymbol{\psi}_{1,k} x_{1,k}^i + \sum_{k=1}^{4} b_{1,k}\boldsymbol{\phi}_{1,k} y_{1,k}^{q_2-1-i}, \;\; \forall i = 0, \ldots, q_2 - 1, \tag{10}$$

where $a_{1,k}$'s and $b_{1,k}$'s are the coefficients to be determined, and $(x_{1,k}, \boldsymbol{\psi}_{1,k})$ $(k = 1, 2, 3, 4)$ are the eigenvalue, left-eigenvector solution pairs of the matrix equations

$$\boldsymbol{\psi}_{1,.}[A_{01} + A_{10}x_{1,.} + A_{21}x_{1,.}^2] = 0, \tag{11}$$

and $(y_{1,k}, \boldsymbol{\phi}_{1,k})$ $(k = 1, 2, 3, 4)$ are the eigenvalue, left-eigenvector solution pairs of the matrix equation

$$\boldsymbol{\phi}_{1,.}[A_{21} + A_{10}y_{1,.} + A_{01}y_{1,.}^2] = 0. \tag{12}$$

The probability π_i, $i \geq q_2$, is given by

$$\pi_i = \sum_{k=1}^{4} a_{2,k}\boldsymbol{\psi}_{2,k}x_{2,k}^{i-q_2}, \quad \forall i \geq q_2, \tag{13}$$

where $(x_{2,k}, \boldsymbol{\psi}_{2,k})$ $(k = 1, 2, 3, 4)$ are the eigenvalue, left-eigenvector solution pairs of the matrix equations

$$\boldsymbol{\psi}_{2,.}[A_{02} + A_{12}x_{2,.} + A_{22}x_{2,.}^2] = 0. \tag{14}$$

Note that $|x_{2,k}| < 1$ for $k = 1, 2, 3, 4$.

The R matrix. Following [**5**], the characteristic matrix polynomial associated to eq. (2) is $Q(x) = A_{02} + xA_{12} + x^2A_{22}$ (note that we use the same notations of [**1**]). We obtain

$$Q(x) = \begin{bmatrix} -(\gamma+\lambda)x & x(\lambda+\gamma x) & 0 & 0 \\ \mu_1 x & -(\gamma+\lambda+\mu_1)x & 0 & x(\lambda+\gamma x) \\ \mu_2 x & 0 & -(\gamma+\lambda+\mu_2)x & x(\lambda+\gamma x) \\ 0 & \mu_2 x & \mu_1 x & \lambda-(\lambda+\mu_1+\mu_2)x \end{bmatrix}. \tag{15}$$

It yields

$$\begin{aligned} Det[Q(x)] &= x^3(x-1)[\lambda((\gamma+\lambda)^2+\gamma\mu_1)(\gamma+\lambda+\mu_2) \\ &\quad -\gamma(\mu_1(\gamma^2+2\lambda^2+\gamma(3\lambda+\mu_1)) + ((\gamma+\lambda)^2+2(\gamma+2\lambda)\mu_1+\mu_1^2)\mu_2 + (\gamma+\lambda+\mu_1)\mu_2^2)x \\ &\quad +\gamma^2\mu_1(\mu_1-\mu_2)x^2] \\ &= x^3(x-1)[\omega_0 - \omega_1 x + \omega_2 x^2] \qquad (16) \end{aligned}$$

where

$$\begin{aligned} \omega_0 &= \lambda((\gamma+\lambda)^2+\gamma\mu_1)(\gamma+\lambda+\mu_2), \\ \omega_1 &= \gamma(\mu_1(\gamma^2+2\lambda^2+\gamma(3\lambda+\mu_1)) + ((\gamma+\lambda)^2+2(\gamma+2\lambda)\mu_1+\mu_1^2)\mu_2 + (\gamma+\lambda+\mu_1)\mu_2^2), \\ \omega_2 &= \gamma^2\mu_1(\mu_1-\mu_2). \end{aligned}$$

As a consequence, $Det[Q(x)]$ has three zero roots $(x_{2,1} = x_{2,2} = x_{2,3} = 0)$, one root equal to 1. In addition,

- $Det[Q(x)]$ has one root $x_{2,4}^* = \frac{\gamma^2\lambda+2\gamma\lambda^2+\lambda^3+\gamma\lambda\mu_2}{\gamma\mu_2(2\gamma+3\lambda+2\mu_2)}$ if $\mu_1 = \mu_2$.
- $Det[Q(x)]$ has two roots $x_{2,4} = \frac{\omega_1-\sqrt{\omega_1^2-4\omega_0\omega_2}}{2\omega_2}$ and $x_{2,5} = \frac{\omega_1+\sqrt{\omega_1^2-4\omega_0\omega_2}}{2\omega_2}$ if $\mu_1 \neq \mu_2$.

Note that the eigenvalues of eq. (14) are the roots of $Det[Q(x)]$. Following the same argument as in [**5**], if the QBD process is ergodic, $Q(x)$ should have four eigenvalues inside the unit circle. As a consequence, $|x_{2,4}^*| < 1$ (for $\mu_1 = \mu_2$) and $|x_{2,4}| < 1$ (for $\mu_1 \neq \mu_2$). In what follows, we also use $x_{2,4}$ to refer to $x_{2,4}^*$ when $\mu_1 = \mu_2$.

It is easy to check that independent left-eigenvectors corresponding to three null-eigenvalues are $\boldsymbol{\psi}_{2,1} = [1, 0, 0, 0]$, $\boldsymbol{\psi}_{2,2} = [0, 1, 0, 0]$,…,$\boldsymbol{\psi}_{2,3} = [0, 0, 1, 0]$.

Let $\boldsymbol{\psi}_{2,4} = [\psi_{2,4,1}, \psi_{2,4,2}, \psi_{2,4,3}, 1]$ be the eigenvector corresponding to $x_{2,4}$. Utilizing $\boldsymbol{\psi}_{2,4}Q(x_{2,4}) = \mathbf{0}$, we get

$$\begin{aligned} \psi_{2,4,1} &= \frac{2\gamma\mu_1\mu_2 + 2\lambda\mu_1\mu_2 + \mu_1^2\mu_2 + \mu_1\mu_2^2}{(\gamma+\lambda+\mu_2)(\gamma^2+2\gamma\lambda+\lambda^2+\gamma\mu_1-\gamma\mu_1 x_{2,4})}, \\ \psi_{2,4,2} &= -\frac{-\gamma^2\mu_2 - 2\gamma\lambda\mu_2 - \lambda^2\mu_2 - \lambda\mu_1\mu_2 - \gamma\mu_2^2 - \lambda\mu_2^2 - \gamma\mu_1\mu_2 x_{2,4}}{(\gamma+\lambda+\mu_2)(\gamma^2+2\gamma\lambda+\lambda^2+\gamma\mu_1-\gamma\mu_1 x_{2,4})}, \\ \psi_{2,4,3} &= \frac{\mu_1}{\gamma+\lambda+\mu 2}. \end{aligned}$$

Following [**5**], we get $R = \Psi^{-1} \cdot diag(0,0,0,x_{2,4}) \cdot \Psi$, where $\Psi = [\boldsymbol{\psi}_{2,1}, \boldsymbol{\psi}_{2,2}, \boldsymbol{\psi}_{2,3}, \boldsymbol{\psi}_{2,4}]^T$. Hence,

$$R = x_4 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \psi_{2,4,1} & \psi_{2,4,2} & \psi_{2,4,3} & 1 \end{bmatrix}. \tag{17}$$

1.3. Finite Orbit Size. The matrix geometric solution for π_i, $i \geq q_2$ is given by

$$\pi_i = \pi_{q2} R_3^{i-q2} + \pi_N R_4^{N-i}, \ \ q_2 \leq i \leq N, \tag{18}$$

where R_3 and R_4 is the minimal non-negative solution of the quadratic matrix equations

$$A_{02} + R_3 A_{12} + R_3^2 A_{22} = 0, \tag{19}$$

$$A_{22} + R_4 A_{12} + R_4^2 A_{02} = 0, \tag{20}$$

respectively.

The alternative expression for π_i, $i \geq q_2$, is given by

$$\pi_i = \sum_{k=1}^{4} a_{2,k} \boldsymbol{\psi}_{2,k} x_{2,k}^{i-q_2} + \sum_{k=1}^{4} b_{2,k} \boldsymbol{\phi}_{2,k} y_{2,k}^{N-i}, \ \ \forall q_2 \leq i \leq N, \tag{21}$$

where $(x_{2,k}, \boldsymbol{\psi}_{2,k})\,(k = 1,2,3,4)$ are the eigenvalue (of least absolute value), left-eigenvector solution pairs of the matrix equations

$$\boldsymbol{\psi}_{2,.}[A_{02} + A_{12} x_{2,.} + A_{22} x_{2,.}^2] = 0, \tag{22}$$

and $(y_{2,k}, \boldsymbol{\phi}_{2,k})\,(k = 1,2,3,4)$ are the eigenvalue (of least absolute value), left-eigenvector solution pairs of the matrix equation

$$\boldsymbol{\phi}_{2,.}[A_{22} + A_{12} y_{2,.} + A_{02} y_{2,.}^2] = 0. \tag{23}$$

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