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Finite source retrial queues with two phase service

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Abstract: This paper is concerned with a single server retrial queue with a finite number of sources with second optional service. This queueing system assumes that each source can generate a primary call to request service when it is free. After the first phase service, the customer will choose the second phase service with probability *p*. Our analysis extends previous work on this topic and we can use method of discrete transformation to get some more general results on the length of the queue, busy period and waiting time. Numerical examples show the influence of key parameters on the performance of the system.

Keywords: retrial queues; finite source; two phase service; discrete transformation; busy period.

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1 Introduction

Queues with repeated attempts are pervasive in some real-life situations where customers may not necessarily enter service immediately because all the servers are not available upon arrival. These customers usually leave the service area and will repeat their demands after some random amount of time called retrial time. Between trials, the blocked customers join a retrial group called ‘orbit’. Examples of retrial queues can be observed in telephone switching systems, call centers management, ethernet systems, mobile communications, etc. For extensive literature on retrial queues, see Falin (1990), Kulkarni and Liang (1997), Falin and Templeton (1997), Artalejo (1999) and Artalejo and Gòmez-Corral (2008).

However, most studies in the retrial queueing literature assume the source size of primary customers to be infinite and thus the flow of primary calls can be modelled by Poisson process. For example, Wu et al. (2005) considered an *M/G/*1 retrial queue with breakdowns and customers may balk or renege at particular times during breakdowns. They derived some useful performance measures of the system such as the idle probability of the system, the mean number of customers in the retrial queue and the expected retrial time. Do et al. (2014a) focused on a retrial queueing system with working vacations and negative customers are Poisson. But, when the customer population is of moderate size, it seems more appropriate that the retrial queueing systems should be studied as a system with finite source of customers. Such a finite source queue is also known as queues with quasi-random input. Retrial queues with quasi-random input are of recent interest in modelling magnetic disk memory systems, telephony, cellular mobile networks, computer networks and localarea networks with non-persistent CSMA/CD protocols (see for example, Falin and Artalejo, 1998). Due to their wide applications in above areas, there has been a rapid growth in the literature on retrial queues with a quasi-random input, readers may refer to Do et al. (2014b), Dragieve (2013, 2014), Roszik et al. (2007), Wang and Li (2009), Wang et al. (2011), Zhang and Wang (2013) and the references therein.

There exists another substantial literature on queueing systems with two-phase service in which the server may provide a second phase service (SPS). Such queueing situations occur in day-to-day life, for example in a manufacturing process all the arriving customers require the main service and only some of them may require the subsidiary service provided by the server. Such kind of systems was firstly discussed by Doshi (1990) and Krishna and Lee (1990) and followed by a series of works such as Choi and Kim (2003), Wang (2004), where the previous results, are extended to include systems allowing server vacations, Bernoulli feedback, server breakdowns, etc. More recently, Tadj et al. (2012) discussed the optimal management policy through a two-phase quorum queueing system. This system is characterised by Bernoulli vacation schedule and an unreliable server with delayed repair. Jain et al. (2013) extended the SPS to multi-optional services and servers' vacations were allowed. Wang and Li (2010) studied an $M^{[X]}/G/1$ queue with second multi-optional service and unreliable server and both of the transient and steady-state solutions for queueing and reliability measures of interest were obtained. One may refer to Bhagat and Jain (2013) for recent advance in this area.

The first investigation of retrial queue with second optional service was done by Kumar et al. (2002). They considered an $M/G/1$ retrial queue with second optional service, where at the first phase of service, the server may push out the customer who is receiving such a service, to start the service of another priority arriving customer. The interrupted customers join a retrial queue and the head of this queue is allowed to conduct a repeated attempt in order to start again his essential service after some random time. Artalejo and Choudhury (2004) provided extensive analysis of such an $M/G/1$ retrial queue under classical retrial policy. They also provided some interesting applications of the model. Wang and Li (2009) studied a single server retrial queue with general retrial times and two-phase service. Choudhury and Kalita (2009) dealt with the steady behaviour of an $M/G/1$ retrial queue which the server provides two phases of heterogeneous service under Bernoulli vacation. Dimitriou and Langaris (2008) considered a two-phase service model with both an ordinary queue and a retrial box receiving all customers who, upon finishing the first phase service (FPS), do not proceed to the second immediately but instead they decide to retry for the second phase later. Atencia and Moreno (2006) investigated discrete time version of Artalejo and Choudhurys model. Wang and Zhao (2007) generalised discrete time version of this model by assuming that the server was subject to starting failures. Recently, Lakshmi and Ramanath (2014a) used the matrix geometric approach to model the $M/M/1$ retrial queue with Bernoulli feedback, impatient customers and two phases of service with server's breakdown and repair. Later on, this paper was extended to a more general model as an $M/G/1$ retrial queue (Lakshmi and Ramanath, 2014b). Bhagat and Jain (2013) studied the queueing measures of the $M^{[x]}/G/1$ retrial queue with batch arrival customers and unreliable server which provides the second optional phase service after the FPS to only those customers who opt for it.

An examination of these papers shows that there is no work on retrial queueing model with a finite number of sources and two-phase service. In view of the growing demands for flexible service in practice, our objective here is to investigate the performance of such a single-server queue with two-phase service simultaneously allowing for finite-source inputs and repeated attempts. It has applications in various fields, for instance, in flexible manufacturing system, there are versatile, multi-functional machines which can perform several types of operations, e.g., lathing, drilling, milling and so forth. Workpieces arrive with different processing requirements, all need the main essential service, whereas some of them may require further particular type of operation after the main essential service. The organisation of the paper is as follows. The model under consideration is described in Section 2. Outside observer's distribution of the server state and the queue length are obtained in Section 3. Section 4 investigates the busy period of the server. In Section 5, the waiting time distribution is discussed and the mean waiting time is given. Finally, in Section 6 we show some numerical examples to illustrate the impact of the parameters on the system performance.

2 Model description

We consider a single-server retrial queue with a finite number of sources of primary customers, where the server provides two phases of heterogeneous service. It is assumed that there are K ($2 \le K < \infty$) sources of primary customers in the system. When a source is free at time t it may generate a primary request for service during interval (t, $t + dt$) with probability αdt. If the server is free at the time of arrival of a primary customer then the customer starts to be served immediately. During the service time the source cannot generate a new primary request. After service the source moves into the free state and it can generate a new primary request. We assume that there is no queue in front of the server, so if the server is busy at time of the arrival of a primary customer, then the source starts generating a Poisson flow of repeated requests with rate μ until it finds the server free. These repeated requests form a virtual retrial orbit in which they will try their luck after some random amount of time in a random manner. As before, after service the source becomes free and can generate a new primary request.

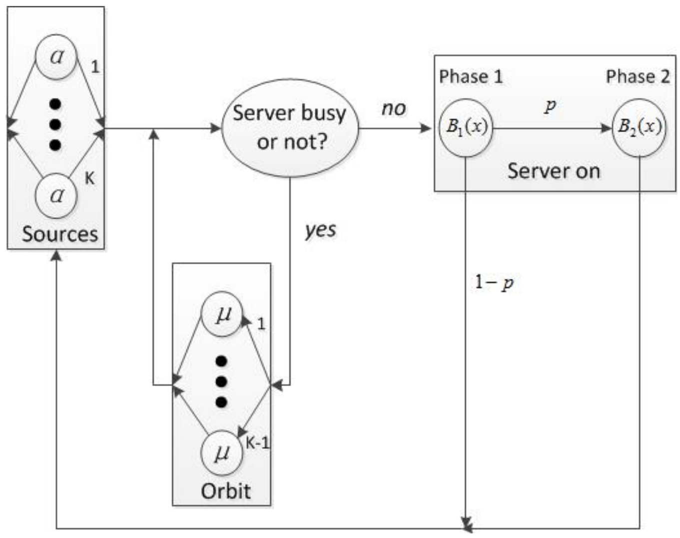
The server provides to each customer two phases of heterogeneous service in succession, where the FPS is followed by the SPS. FPS is needed by all arriving customers, but after FPS the customer will leave the server with probability p or choose the SPS with probability $\overline{p} = 1 - p$. The service discipline is assumed to be FCFS. It is assumed that the service time random variable B_i(i = 1, 2) of the i^{th} phase of service follows a general probability law with distribution function (d.f.) $B_i(x)$, Laplace-Stieltjes transform (LST) $\beta_i(s)$, hazard rate function $b_i(x)$, and n^{th} moments

$$\int_0^\infty x^n dB_i(x) = \beta_{i,n},$$

for $i \in \{1, 2\}$, respectively. We assume that primary requests, repeated attempts and each phase service times are mutually independent. Figure 1 illustrates the dynamics of the queueing system.

The state of the system at time t can be described by the process ($C(t)$, $N(t)$), where $C(t)$ is 0, 1 or 2 according to whether the server is free, offering FPS or offering SPS at time t, and $N(t)$ is the number of sources of repeated customers at time t. In order to work with a Markov process in the case $C(t) > 0$ we introduce two supplementary variables: if $C(t) = 1$, then ξ_1 represents the remaining service time in FPS and if $C(t) = 2$, ξ_2 corresponds to the remaining service time of the customer being served in SPS. Obviously, the situation $C(t) = 0$, $N(t) = K$ is impossible and thus the state space of the process ($C(t)$, $N(t)$) is the set $\{0, 1, 2\} \times \{0, 1, 2,\ldots,K - 1\}$.

Figure 1 Dynamics of the queueing system



3 Outside observer's distribution of the server state and the queue length

For the process $(C(t), N(t))$, we define:

$$p_{0n}(t) = P\{C(t) = 0, N(t)\}, \quad 0 \le n \le K-1,$$

$$P_{1n}(t, x) = \frac{d}{dx} P\{C(t) = 1, \xi_1(t) \le x, N(t) = n\}, \quad 0 \le n \le K-1,$$

$$P_{2n}(t, y) = \frac{d}{dy} P\{C(t) = 2, \xi_2(t) \le y, N(t) = n\}, \quad 0 \le n \le K-1.$$

Then, following a general way by using the method of supplementary variables, we find that the limiting probabilities as $t \to \infty$ satisfy the equations of statistical equilibrium:

$$\big((K-n)\alpha + n\mu\big) p_{0n} = p\int_0^\infty p_{1n}(x) b_1(x) dx + \int_0^\infty p_{2n}(y) b_2(y) dy, \quad 0 \le n \le K-1, \tag{1}$$

$$p'_{1n}(x) = -\big((K-n-1)\alpha + b_1(x)\big) p_{1n}(x) + (K-n)\alpha p_{1,n-1}(x), \quad 0 \le n \le K-1, \tag{2}$$

$$p'_{2n}(y) = -\big((K-n-1)\alpha + b_2(y)\big) p_{2n}(y) + (K-n)\alpha p_{2,n-1}(y), \quad 0 \le n \le K-1, \tag{3}$$

with boundary conditions:

$$p_{1n}(0) = (K-n)\alpha p_{0n} + (n+1)\mu p_{0,n+1}, \quad 0 \le n \le K-1, \tag{4}$$

$$p_{2n}(0) = \overline{p}\int_0^\infty p_{1n}(x) b_1(x) dx, \quad 0 \le n \le K-1, \tag{5}$$

where $p_{0,K} = p_{1,-1}(x) = p_{2,-1}(y) = 0$.

It is convenient to solve such difference equations with the help of so-called discrete transformations (see Falin and Artalejo, 1998):

$$q_m = \sum_{n=1}^{K-1-m} \binom{K-1-n}{m} p_n, \tag{6}$$

and the inverse transformation is given by

$$p_n = \sum_{m=1}^{n} (-1)^m \binom{K-1-n+m}{m} q_{K-1-n+m}, \quad 0 \le n \le K-1. \tag{7}$$

Let q_{0m}, $q_{1m}(x)$, $q_{2m}(y)$, q_{1m}, q_{2m} be the images of sequences p_{0n}, $p_{1n}(x)$, $p_{2n}(y)$, p_{1n}, p_{2n}, respectively, under discrete transformation (6). So the equations (1)–(5) can be rewritten for $0 \le m \le K-1$ as:

$$\begin{aligned}\big((K-m-1)\mu+(m+1)\alpha\big)q_{0,m} + (m+1)(\alpha-\mu)q_{0,m+1} = p\int_0^\infty q_{1,m}(x)b_1(x)dx \\ +\int_0^\infty q_{2,m}(y)b_2(y)dy,\end{aligned} \tag{8}$$

$$q'_{1m}(x) = -\big(m\alpha + b_1(x)\big)q_{1,m}(x), \tag{9}$$

$$q'_{2m}(y) = -\big(m\alpha + b_2(y)\big)q_{2m}(y), \tag{10}$$

$$\begin{aligned}q_{1m}(0) &= \big((m+1)\alpha + (K-2m-1)\mu\big)q_{0,m} \\ &\quad +(K-m)\mu q_{0,m-1} + (m+1)(\alpha-\mu)q_{0,m+1},\end{aligned} \tag{11}$$

$$q_{2m}(0) = \overline{p}\int_0^\infty q_{1,m}(x)b_1(x)dx, \tag{12}$$

where $q_{0,K} = q_{0,-1} = 0$. Note by normalisation condition, $q_{00} + q_{10} + q_{20} = 1$.

By equations (9) and (10), we have

$$q_{1m}(x) = q_{1m}(0)\big(1-B_1(x)\big)\exp(-m\alpha x), \quad 0 \le m \le K-1, \tag{13}$$

$$q_{2,m}(x) = q_{2,m}(0)\big(1-B_2(x)\big)\exp(-m\alpha x), \quad 0 \le m \le K-1. \tag{14}$$

By virtue of equations (13) and (14), the equation (8) can be rewritten as

$$\begin{aligned}\big[\big((K-m-1)\mu+(m+1)\alpha\big)\big]q_{0,m} + (m+1)(\alpha-\mu)q_{0,m+1} \\ = q_{1,m}(0)\beta_1(m\alpha)p + q_{2,m}(0)\beta_2(m\alpha),\end{aligned} \quad 0 \le m \le K-1. \tag{15}$$

By (12), we have

$$\begin{aligned}\big[(K-m-1)\mu+(m+1)\alpha\big]q_{0,m} + (m+1)(\alpha-\mu)q_{0,m+1} \\ = q_{1,m}(0)\beta_1(m\alpha)\big(p+\overline{p}\beta_2(m\alpha)\big),\end{aligned} \quad 0 \le m \le K-1. \tag{16}$$

With the help of (13), (14), we integrate $q_{1,0}(x)$ and $q_{2,0}(y)$ from 0 to ∞:

$$q_{10} = \int_0^\infty q_{10}(x)dx = q_{10}(0)\int_0^\infty \big(1-B_1(x)\big)dx = q_{10}(0)v_1^{-1},$$

$$q_{20} = \int_0^\infty q_{20}(y)dy = q_{20}(0)\int_0^\infty \left(1 - B_2(y)\right)dy = q_{20}(0)v_2^{-1},$$

where

$$v_1^{-1} = \int_0^\infty \left(1 - B_1(x)\right)dx, \quad v_2^{-1} = \int_0^\infty \left(1 - B_2(y)\right)dy.$$

From the equation

$$q_{00} + q_{10} + q_{20} = q_{00} + q_{10}(0)v_1^{-1} + q_{2,0}(0)v_2^{-1} = 1,$$

the following result can be derived:

$$\left(v_1v_2 + \left(v_2 + \bar{p}v_1\right)\left(\alpha + (K-1)\mu\right)\right)q_{00} + \left(v_2 + \bar{p}v_1\right)(\alpha - \mu)q_{01} = v_1v_2. \tag{17}$$

Eliminating $q_{1,m}(0)$ from (11) and (16), we get:

$$\begin{aligned}
&\Big[\left((k-m-1)\mu + (m+1)\alpha\right)\left(1 - \beta_1(m\alpha)\left(p + \bar{p}\beta_2(m\alpha)\right)\right) \\
&\quad + m\mu\beta_1(m\alpha)\left(p + \bar{p}\beta_2(m\alpha)\right)\Big]q_{0m} \\
&\quad - (k-m)\mu\beta_1(m\alpha)\left(p + \bar{p}\beta_2(m\alpha)\right)q_{0m-1} \\
&\quad + (m+1)(\alpha - \mu)\left(1 - \beta_1(m\alpha)\left(p + \bar{p}\beta_2(m\alpha)\right)\right)q_{0m+1} = 0,\ 0 \le m \le K-1.
\end{aligned} \tag{18}$$

From (17) and (18), variables q_{0m}, $0 \le m \le K - 1$ can be easily determined numerically with the help of the following procedure: firstly, it is clear from (18) that all variables q_{0m}, $0 \le m \le K - 1$ are proportional to $q_{0,K-1}$, so we let $q_{0,m} = C_m q_{0,K-1}$, $0 \le m \le K - 2$. The coefficients C_m can be recursively computed from (18) by putting $q_{0,K-1} = 1$. Then $q_{0,K-1}$ can be calculated from (17):

$$q_{0,K-1} = v_1v_2\Big[\left(v_1v_2 + \left(\bar{p}v_1 + v_2\right)\left(\alpha + (K-1)\mu\right)\right)C_0 + \left(v_2 + \bar{p}v_1\right)(\alpha - \mu)C_1\Big]^{-1}.$$

Then we can calculate variables $q_{1m}(0)$, and then $q_{1m}(x)$, $q_{2m}(y)$ from (11), (13) and (14). After that, q_{1m}, q_{2m} can be also derived. With the help of inverse transformations (7), we may determine that original probabilities $p_{0,n}$, $p_{1,n}(x)$, $p_{2,n}(y)$, $p_{1,n}$ and $p_{2,n}$, respectively.

Because we are interested only in macro-characteristics of the system rather than in probabilities $p_{i,n}$, then we need not use inverse transformation in order to get the probabilities $p_{i,n}$. In the following, we give main performance characteristics of the system which are expressed in terms of variables $q_{i,n}$.

1 the probability that the server is idle:

$$p_0 = \sum_{n=1}^{K-1} p_{0n} = q_{00}. \tag{19}$$

2 the probability that the server is busy:

$$p_B = 1 - q_{00} = q_{10} + q_{20}. \tag{20}$$

3 the mean number of sources of repeated requests:

$$N = \sum_{n=0}^{K-1} n\left(p_{0n} + p_{1n} + p_{2n}\right) = K - 1 - \left(q_{01} + q_{11} + q_{21}\right)$$
$$= K - \frac{v_1 v_2 + \alpha\left(v_2 + \bar{p} v_1\right)}{\alpha(\bar{p} v_1 + v_2)} p_B. \tag{21}$$

4 the mean rate of generation of primary arrivals:

$$\bar{\lambda} = \alpha E\left[K - N(t) - \left(1 - P\left(C(t) = 0\right)\right)\right] = \frac{v_1 v_2}{\bar{p} v_1 + v_2} p_B. \tag{22}$$

5 the mean waiting time $E(W)$ can be easily obtained by using Little's formula:

$$E(W) = \left(\bar{\lambda}\right)^{-1} N = K \frac{\bar{p} v_1 + v_2}{v_1 v_2 P_B} - \alpha^{-1} - \frac{\bar{p} v_1 + v_2}{v_1 v_2}. \tag{23}$$

4 Busy period

Assume that all sources are free at time $t_0 = 0$ and one of them just generates a request for service which initiates a busy period. The busy period ends at the first service completion epoch at which $(C(t), N(t))$ returns to the state (0, 0). The busy period consists of the FPS periods, the SPS periods and periods during which the server is free and there are sources of repeated calls in the system. The length of the busy periods will be denoted by L, its distribution function $P(L \leq t)$ by $\Pi(t)$ and its LST by $\pi(s)$.

Let a busy period starts at time $t_0 = 0$. Define:

$$P_{0n}(t) = P\left\{L > t, C(t) = 0, N(t) = n\right\}, \quad 1 \leq n \leq K - 1, \tag{24}$$

$$P_{1n}(t, x) = P\left\{L > t, C(t) = 1, N(t) = n, \xi_1(t) \in (x, x + dx)\right\}, \quad 0 \leq n \leq K - 1, \tag{25}$$

$$P_{2,n}(t, y) = P\left\{L > t, C(t) = 2, \xi_2(t) \in (y, y + dy)\right\}, \quad 0 \leq n \leq K - 1, \tag{26}$$

$$P_{1,n}(t) = P\left\{L > t, C(t) = 1, N(t) = n\right\}, \quad 0 \leq n \leq K - 1, \tag{27}$$

$$P_{2,n}(t) = P\left\{L > t, C(t) = 2, N(t) = n\right\}, \quad 0 \leq n \leq K - 1. \tag{28}$$

By routine procedure, Kolmogorov differential equations that govern the dynamics of these taboo probabilities are given by:

$$\frac{dP_{0n}(t)}{dt} = -\left[(K - n)\alpha + n\mu\right] P_{0n}(t) + p\int_0^\infty P_{1n}(t, x) b_1(x) dx + \int_0^\infty P_{2n}(t, y) b_2(y) dy, \quad 1 \leq n \leq K - 1, \tag{29}$$

$$\frac{d\Pi(t)}{dt} = p\int_0^\infty P_{10}(t, x) b_1(x) dx + \int_0^\infty P_{20}(t, y) b_2(y) dy, \tag{30}$$

$$\frac{\partial P_{1n}(t,x)}{\partial t} = -\left[(K-n-1)\alpha + b_1(x) + \frac{\partial}{\partial x}\right] P_{1n}(t,x) \quad 0 \le n \le K-1, \tag{31}$$
$$+(K-n)\alpha P_{1,n-1}(t,x),$$

$$\frac{\partial P_{2n}(t,y)}{\partial t} = -\left[(K-n-1)\alpha + b_2(y) + \frac{\partial}{\partial y}\right] P_{2,n}(t,y) \quad 0 \le n \le K-1, \tag{32}$$
$$+(K-n)\alpha P_{2,n-1}(t,y),$$

$$P_{1,n}(t,0) = (K-n)\alpha P_{0,n}(t) + (n+1)\mu P_{0,n+1}(t), \quad 0 \le n \le K-1, \tag{33}$$

$$P_{2,n}(t,0) = \overline{p}\int_0^\infty P_{1n}(t,x) b_1(x)dx, \quad 0 \le n \le K-1, \tag{34}$$

where $P_{00}(t) = P_{0K}(t) = P_{1,-1}(t, x) = P_{2,-1}(t, y) = P_{0n}(0) = P_{2n}(0, y) = 0$, $P_{10}(0, 0) = 1$. $P_{1n}(0, x) = \delta(x)\delta_{n0}$, $\delta(x)$ is Dirac delta function and δ_{ij} is Kronecker's delta.

Define:

$$\varphi_{0,n}(s) = \int_0^\infty e^{-st} P_{0,n}(t)dt, \quad \varphi_{1,n}(s,x) = \int_0^\infty e^{-st} P_{1,n}(t,x)dt,$$
$$\varphi_{2,n}(s,y) = \int_0^\infty e^{-st} P_{2,n}(t,y)dt.$$

So equations (29)–(34) can be rewritten as:

$$\left(s + (K-n)\alpha + n\mu\right)\varphi_{0n}(s) = p\int_0^\infty \varphi_{1n}(t,x) b_1(x)dx \quad 0 \le n \le K-1, \tag{35}$$
$$+\int_0^\infty \varphi_{2,n}(t,y) b_2(y)dy,$$

$$\pi(s) = p\int_0^\infty \varphi_{10}(s,x) b_1(x)dx + \int_0^\infty \varphi_{2,0}(s,y) b_2(y)dy, \tag{36}$$

$$\frac{\partial \varphi_{1n}(s,x)}{\partial t} = -\left(s + (K-n-1)\alpha + b_1(x)\right)\varphi_{1n}(s,x) \tag{37}$$
$$+(K-n)\alpha\varphi_{1n-1}(s,x) + \delta(x)\delta_{n0},$$

$$\frac{\partial \varphi_{2n}(s,y)}{\partial y} = -\left(s + (K-n-1)\alpha + b_2(y)\right)\varphi_{2,n}(s,y) \quad 0 \le n \le K-1, \tag{38}$$
$$+(K-n)\alpha\varphi_{2n-1}(s,y),$$

$$\varphi_{1n}(s,0) = (K-n)\alpha\varphi_{0,n}(s) + (n+1)\mu\varphi_{0n+1}(s), \quad 0 \le n \le K-1, \tag{39}$$

$$\varphi_{2n}(s,0) = \int_0^\infty \varphi_{1n}(s,x) b_1(x)\overline{p}dx, \quad 0 \le n \le K-1, \tag{40}$$

We use the discrete transformations method (6) once more, denote by $\psi_{0,n}(s)$, $\psi_{1,n}(s, x)$, $\psi_{2,n}(s, y)$ the images of sequences $\varphi_{0,n}(s)$, $\varphi_{1,n}(s, x)$, $\varphi_{2,n}(s, y)$. So the equations (35)–(40) can be rewritten as:

$$\begin{aligned}&\big(s+(K-m-1)\mu+(m+1)\alpha\big)\psi_{0,m}(s)+(m+1)(\alpha-\mu)\psi_{0,m+1}(s)\\&\quad=\int_0^\infty \psi_{1,m}(s,x)b_1(x)p\,dx \qquad 0\le m\le K\le 1,\\&\quad+\int_0^\infty \psi_{2,m}(s,y)b_2(y)dy-\binom{K-1}{m}\pi(s),\end{aligned} \tag{41}$$

$$\frac{\partial}{\partial x}\psi_{1,m}(s,x)=-\big(s+m\alpha+b_1(x)\big)\psi_{1,m}(s,x)+\binom{K-1}{m}\delta(x),\quad 0\le m\le K-1, \tag{42}$$

$$\frac{\partial}{\partial y}\psi_{2,m}(s,y)=-\big(s+m\alpha+b_2(y)\big)\psi_{2,m}(s,y),\quad 0\le m\le K-1, \tag{43}$$

$$\begin{aligned}\psi_{1m}(s,0)&=\big((m+1)\alpha+(K-2m-1)\mu\big)\psi_{0,m}(s)\\&\quad+(K-m)\mu\psi_{0,m-1}(s)+(m+1)(\alpha-\mu)\psi_{0,m+1}(s),\end{aligned}\qquad 0\le m\le K-1, \tag{44}$$

$$\psi_{2,m}(s,0)=\int_0^\infty \psi_{1,m}(s,x)b_1(x)\overline{p}\,dx,\quad 0\le m\le K-1, \tag{45}$$

where $\psi_{0,-1}(s)=\psi_{0,K-1}(s)=\psi_{0,K}(s)=0$.

With the help of (42) we have

$$\psi_{1,m}(s,x)=\left[\psi_{1,m}(s,0)+\binom{K-1}{m}\right]\exp\{-(s+m\alpha)x\}\big(1-B_1(x)\big), \tag{46}$$

From (43) we get

$$\psi_{2m}(s,y)=\psi_{2m}(s,0)\exp\{-(s+m\alpha)y\}\big(1-B_2(y)\big). \tag{47}$$

By (46), we can rewrite (45) as:

$$\psi_{2,m}(s,0)=\overline{p}\left[\psi_{1,m}(s,0)+\binom{K-1}{m}\right]\beta_1(s+m\alpha),\quad 0\le m\le K-1, \tag{48}$$

Equation (47) can be rewritten as:

$$\psi_{2,m}(s,y)=\overline{p}\left[\psi_{1,m}(s,0)+\binom{K-1}{m}\right]\exp\{-(s+m\alpha)y\}\big(1-B_2(y)\big)\beta_1(s+m\alpha),$$

and equation (41) can be rewritten as:

$$\begin{aligned}&\big(s+(K-m-1)\mu+(m+1)\alpha\big)\psi_{0,m}(s)+(m+1)(\alpha-\mu)\psi_{0,m+1}(s)\\&\quad=\beta_1(s+m\alpha)\left(\psi_{1,m}(s,0)+\binom{K-1}{m}\right)\big(p+\overline{p}\beta_2(s+m\alpha)\big)-\binom{K-1}{m}\pi(s).\end{aligned}\qquad 0\le m\le K-1 \tag{49}$$

Eliminating $\psi_{1m}(s,0)$ by using (44) and (49), we get

$$\mu_m(s)\psi_{0,m}(s)+v_m(s)_{0,m+1}(s)-w_m(s)\psi_{0,m-1}(s)$$
$$=\binom{K-1}{m}\Big[\big(p+\overline{p}\beta_2(s+m\alpha)\big)\beta_1(s+m\alpha)-\pi(s)\Big], \tag{50}$$

where

$$\mu_m(s)=s+\big((K-m-1)\mu+(m+1)\alpha\big)\Big(1-\big(p+\overline{p}\beta_2(s+m\alpha)\big)\beta_1(s+m\alpha)\Big)$$
$$+m\mu\big(p+\overline{p}\beta_2(s+m\alpha)\big)\beta_1(s+m\alpha),$$

$$v_m(s)=(m+1)(\alpha-\mu)\Big(1-\big(p+\overline{p}\beta_2(s+m\alpha)\big)\beta_1(s+m\alpha)\Big),$$

$$w_m(s)=(K-m)\mu\big(p+\overline{p}\beta_2(s+m\alpha)\big)\beta_1(s+m\alpha).$$

Letting $m = K-1, K-2,\ldots, 0$, then from (50) we have

$$\psi_{0,m}(s)=C_m(s)\pi(s)+D_m(s).$$

Since $\psi_{0,K-1}(s) = 0$, we have

$$C_{K-1}(s)=0,\quad D_{K-1}(s)=0,$$
$$C_{K-2}(s)=\Big(\mu\beta_1\big(s+(K-1)\alpha\big)\big(p+\overline{p}\beta_2\big(s+(K-1)\alpha\big)\big)\Big)^{-1}, \tag{51}$$
$$D_{K-2}(s)=\mu^{-1}.$$

and

$$\mu_m(s)C_m(s)+v_m(s)C_{m+1}(s)-w_m(s)C_{m-1}(s)=-\binom{K-1}{m},\quad 1\le m\le K-2,$$
$$\mu_m(s)D_m(s)+v_m(s)D_{m+1}(s)-w_m(s)D_{m-1}(s) \tag{52}$$
$$=\binom{K-1}{m}\beta_1\big(s+m\alpha\big)\big(p+\overline{p}\beta_2(s+m\alpha)\big),\qquad 1\le m\le K-2.$$

Letting $m = 0$ in (50) it follows that

$$\pi(s)=\Big[\big(p+\overline{p}\beta_2(s)\big)\beta_1(s)-\Big(s+\big((K-1)\mu+\alpha\big)\Big(1-\big(p+\overline{p}\beta_2(s)\big)\beta_1(s)\Big)D_0(s)$$
$$-(\alpha-\mu)\Big(1-\big(p+\overline{p}\beta_2(s)\big)\beta_1(s)\Big)D_1(s)\Big)\Big]\times\Big\{1+\Big(s+\big((K-1)\mu+\alpha\big)$$
$$\Big[1-\big(p+\overline{p}\beta_2(s)\big)\beta_1(s)\Big]C_0(s)+(\alpha-\mu)\Big[1-\big(p+\overline{p}\beta_2(s)\big)\beta_1(s)\Big]C_1(s)\Big)\Big\}^{-1}.$$

Differentiating $\pi(s)$ with respect to s at the point $s = 0$ we get the following formula:

$$\pi'(0)=\big(\overline{p}\beta_2'(0)+\beta_1'(0)\big)+\Big(\big((K-1)\mu+\alpha\big)\big(\overline{p}\beta_2'(0)+\beta_1'(0)\big)-1\Big)\big(C_0(0)+D_0(0)\big)$$
$$+(\alpha-\mu)\big(\overline{p}\beta_2'(0)+\beta_1'(0)\big)\big(C_1(0)+D_1(0)\big).$$

Therefore

$$E[L] = -\pi'(0) = \left(\overline{p}\beta_{21} + \beta_{11}\right) + \left(((K-1)\mu + \alpha)\left(\overline{p}\beta_{21} + \beta_{11}\right) + 1\right)\left(C_0(0) + D_0(0)\right)$$
$$+(\alpha - \mu)(\overline{p}\beta_{21} + \beta_{11})\left(C_1(0) + D_1(0)\right),$$

where $\beta_{11} = -\beta_1'(0), \beta_{21} = \beta_2'(0),$ and the parameters $D_0(0)$, $D_1(0)$, $C_0(0)$, $C_1(0)$ can be obtained by solving equations (51)–(52).

5 Waiting time

To evaluate the quality of service of customers it is essential to know the state of the system at the time when a particular source generates a new primary request. We denote this arriving customer's distribution by π_{in}, which can be easily related to the outside observer's distribution p_{in}. With this goal let us characterise the state of the system with the help of the vector $x = (x_1,\dots, x_K)$ defined in Falin and Artalejo (1998), that is

$$x_i = \begin{cases} 0, & \text{if } ith \text{ source is free;} \\ s, & \text{if } ith \text{ source is served;} \qquad 1 \le i \le K. \\ r, & \text{if } ith \text{ source is in retrial orbit.} \end{cases}$$

So,

$$N(x) = \sum_{i=1}^{K} I\left(x_i = r\right)$$

denotes the number of customers in the retrial orbit.

For any microscopic state x denote by $C(x)$ the number of served calls, and by $N(x)$ the number of sources of repeated calls. Let X_{in} be the set of all microscopic states x such that $C(x) = i$, $N(x) = n$; $i = 0, 1, 2$; $0 \le n \le K - 1$. It is easy to see that the cardinality of the set X_{0n} is

$$\binom{K}{n},$$

the cardinality of the set X_{1n} is

$$K\bullet\binom{K-1}{n}$$

and the cardinality of the set X_{2n} is

$$K\bullet\binom{K-1}{n}.$$

Due to the symmetry all states $x \in X_{0n}$ have the same probability p_{in}^*. On the other hand, the probability of the set X_{in} is outside observer's probability p_{in}. Thus, we have

$$p_{0n}^* = p_{0n}\binom{K}{n}^{-1}, \quad p_{1n}^* = p_{1n}K^{-1}\binom{K-1}{n}^{-1}, \quad p_{2n}^* = p_{2n}K^{-1}\binom{K-1^{-1}}{n}.$$

Now fix some source i_0, and denote by $\hat{p}_{0n}$ the probability that in steady state this source is free, the server is free and there are n sources of repeated calls. It is clear that this event is formed by states $x \in X_{0n}$ such that $x_{i0} = 0$. The total number of such states is

$$\binom{K-1}{n}.$$

Thus, we obtain

$$\hat{p}_{0n} = \binom{K-1}{n} p^*_{0n} = K^{-1}(K-n)p_{0n}.$$

Similarly, for the probability $\hat{p}_{1n}$ that in steady state the fixed source is free, the server is busy and there are n sources of repeated calls, and the probability $\hat{p}_{2n}$ that in steady state the fixed source is free, the server is under repair and there are n sources of repeated calls, we have

$$\hat{p}_{1n} = (K-1)\binom{K-2}{n} p^*_{1n} = K^{-1}(K-n-1)p_{1n},$$

$$\hat{p}_{2n} = (K-1)\binom{K-2}{n} p^*_{2n} = K^{-1}(K-n-1)p_{2n},$$

Therefore, the stationary probability that the fixed source is free is equal to

$$\hat{p} = \sum_{n=0}^{K-1}\left(\hat{p}_{0n} + \hat{p}_{1n} + \hat{p}_{2n}\right) = (K\alpha)^{-1}\overline{\lambda},$$

and the conditional distribution of the system state provided the fixed source i, is free is given by

$$\pi_{0n} = \tilde{p}_{0n} = (\hat{p})^{-1}\hat{p}_{0n} = (\overline{\lambda})^{-1}(K-n)\alpha p_{0n}, \tag{53}$$

$$\pi_{1n} = \tilde{p}_{1n} = (\hat{p})^{-1}\hat{p}_{1n} = (\overline{\lambda})^{-1}(K-n-1)\alpha p_{1n}, \tag{54}$$

$$\pi_{2n} = \tilde{p}_{2n} = (\hat{p})^{-1}\hat{p}_{2n} = (\overline{\lambda})^{-1}(K-n-1)\alpha p_{2n}, \tag{55}$$

Let P_B be the probability that the server is occupied, i.e., the blocking probability.

1 when $\alpha \neq = \mu$:

$$\begin{aligned} P_B &= \sum_{n=0}^{K-1}(\pi_{1n} + \pi_{2n}) = \left(\overline{\lambda}\right)^{-1}\alpha\sum_{n=0}^{K-1}(K-n-1)(p_{1n} + p_{2n}) \\ &= \left(\overline{\lambda}\right)^{-1}\alpha(q_{11} + q_{21}) = \left(\overline{\lambda}\right)^{-1}\left((K-1)\mu q_{00} - \mu q_{01}\right) \\ &= \left(v_1 v_2 P_B(\alpha-\mu)\right)^{-1}\mu\left[\left(\overline{p}v_1 + v_2\right)K\alpha - \left(v_1 v_2 + K\alpha\left(\overline{p}v_1 + v_2\right)\right)p_B\right] \end{aligned}$$

2 when $\alpha = \mu$: with the help of equations (11), (16) and (17) we have

$$P_B = \frac{(K-1)\left(1-\beta_1(\alpha)\left(p+\overline{p}\beta_2(\alpha)\right)\right)}{K-(K-1)\left(\beta_1(\alpha)\left(p+\overline{p}\beta_2(\alpha)\right)\right)}.$$

The mean of waiting time can be derived from the above results. Moreover, the waiting time distribution can be derived in the light of the approach developed in Falin and Templeton (1997). For convenience, in the following we only consider the special case when the service time is exponentially distributed. To this end, we assume that $B_1(x) = 1 - e^{-x}$, $B_2(y) = 1 - e^{-y}$. Under this assumption, we give the balance equations as follows.

$$\left((K-n)\alpha + n\mu\right) p_{0n} = p_{1n}p + p_{2n}, \tag{56}$$

$$\left((K-n-1)\alpha + 1\right) p_{1n} = (K-n)\alpha p_{0n} + (n+1)\mu p_{0,n+1} + (K-n)\alpha p_{1,n-1}, \tag{57}$$

$$\left((K-n-1)\alpha + 1\right) p_{2n} = (K-n)\alpha p_{2,n-1} + \overline{p}p_{1n}. \tag{58}$$

Assume that at time $t = 0$, there are n sources of repeated calls and i customers in service, $1 \le n \le K-1$, $i = 0, 1, 2$. Mark one of the customers in orbit and denote by $f_{in}(t)$ the probability that by the time t this customer is not served yet, i.e., the residual waiting time of the tagged customer, τ_{in}, is greater than t. In terms of these probabilities the complementary waiting time distribution function $F(t)$ can be expressed as follows:

$$F(t) = \sum_{n=1}^{K-1}\left(\pi_{1,n-1}f_{1n}(t) + \pi_{2,n-1}f_{2n}(t)\right). \tag{59}$$

Theorem 1. The LST $W(s)$ of the virtual waiting time, W, is given by

$$W(s) = 1 - \left(\overline{\lambda}\right)^{-1} s \sum_{i=0}^{2}\sum_{n=0}^{K-1} np_{in}\tau_{in}(s),$$

where $\tau_{in}(s)$ denoted the LST of the conditional waiting times, τ_{in}. The n^{th} moment of the waiting time W are given by:

$$E\left[W^n\right] = \delta_{n0} + \left(\overline{\lambda}\right)^{-1} n \sum_{i=0}^{2}\sum_{j=0}^{K-1} jp_{ij}E\left[\tau_{ij}^{n-1}\right].$$

The mean waiting time is given by:

$$E[W] = \left(\overline{\lambda}\right)^{-1} \sum_{i=0}^{2}\sum_{j=0}^{K-1} jp_{ij}$$

Proof. To prove the theorem we introduce an auxiliary Markov process $\zeta(t)$ with the state space $\{0, 1, 2\} \times \{1, 2, \ldots, K-1\} \cup \{(1, 0)\}$ and the following rates of transition:

$$q_{(1,n)\to(j,m)} = \begin{cases} p, & (j,m) = (0,n); \\ (K-n-1)\alpha, & (j,m) = (1,n+1); \\ \overline{p}, & (j,m) = (2,n). \end{cases}$$

$$q_{(0,n)\to(j,m)} = \begin{cases} \mu, & (j,m)=(1,0); \\ (n-1)\mu, & (j,m)=(1,n-1); \\ (K-n)\alpha, & (j,m)=(1,n). \end{cases}$$

$$q_{(2,n)\to(j,m)} = \begin{cases} 1, & (j,m)=(0,n); \\ (K-n-1)\alpha, & (j,m)=(2,n+1); \end{cases}$$

and

$$f_{in}(t) = P\left\{\zeta(t) \neq (1,0) \middle| \zeta(0) = (i,n)\right\}.$$

Hence

$$f'_{0n}(t) = -\big((K-n)\alpha + n\mu\big) f_{0n}(t) + (n-1)\mu f_{1,n-1}(t) + (K-n)\alpha f_{1n}(t), \tag{60}$$

$$f'_{1n}(t) = -\big((K-n-1)\alpha + 1\big) f_{1n}(t) + p f_{0n}(t) + (K-n-1)\alpha f_{1,n+1}(t) + \overline{p} f_{2,n}(t), \tag{61}$$

$$f'_{2n}(t) = -\big(1 + (K-n-1)\alpha\big) f_{2,n}(t) + f_{0,n}(t) + (K-n-1)\alpha f_{2,n+1}(t), \tag{62}$$

Our main objective is to develop a recursive scheme for computing any arbitrary moment of the waiting time process. To that end we now denote the LST of $f_{ij}(t)$ by $\phi_{ij}(s)$, so equations (60)–(62) become:

$$-1 + s\phi_{0n}(s) = -\big((K-n)\alpha + n\mu\big)\phi_{0,n}(s) + (n-1)\mu\phi_{1,n-1}(s) + (K-n)\alpha\phi_{1,n}(s), \tag{63}$$

$$\begin{aligned} -1 + s\phi_{1n}(s) = &-\big((K-n-1)\alpha + 1\big)\phi_{1n}(s) + p\phi_{0,n}(s) \\ &+(K-n-1)\alpha\phi_{1,n+1}(s) + \overline{p}\phi_{2n}(s), \end{aligned} \tag{64}$$

$$-1 + s\phi_{2,n}(s) = -\big(1 + (K-n-1)\alpha\big)\phi_{2,n}(s) + \phi_{0,n}(s) + (K-n-1)\alpha\phi_{2,n+1}(s), \tag{65}$$

Now we obtain a useful formula for the LST of the virtual waiting time, which can be thought of as generalisation of Little's formula for the mean waiting time. With this goal multiply equations (63)–(65) by np_{0n}, np_{1n}, np_{2n}, respectively, replace $((K-n)\ \alpha + n\mu)p_{0n}$, $((K-n-1)\ \alpha + 1 + \lambda)p_{1n}$ and $((k-n-1)\ \alpha + 1)p_{2n}$ in the right-hand sides of the resulting equations with the help of the equations of statistical equilibrium equations (56)–(58), and sum these three equations. After some algebra we get:

$$-N + s\sum_{i=0}^{2}\sum_{n=0}^{K-1} np_{in}\phi_{in}(s) = -\sum_{n=0}^{K-1}\big((K-n-1)\alpha\, p_{1n}\phi_{1,n+1}(s) + (K-n-1)\alpha\, p_{2n}\phi_{2,n+1}(s)\big),$$

that is

$$N - s\sum_{i=0}^{2}\sum_{n=0}^{K-1} np_{in}\phi_{in}(s) = \sum_{n=0}^{K-1}\big((K-n-1)\alpha\, p_{1n}\phi_{1n+1}(s) + (K-n-1)\alpha\, p_{2n}\phi_{2n+1}(s)\big).$$

Introducing the LST $W(s)$ of the virtual waiting time, W, and LST $\tau_{in}(s)$ of the conditional waiting times, τ_{in}, and we can rewrite this equation as follows:

$$W(s)=1-\left(\bar{\lambda}\right)^{-1}s\sum_{i=0}^{2}\sum_{n=0}^{K-1}np_{in}\tau_{in}(s).$$

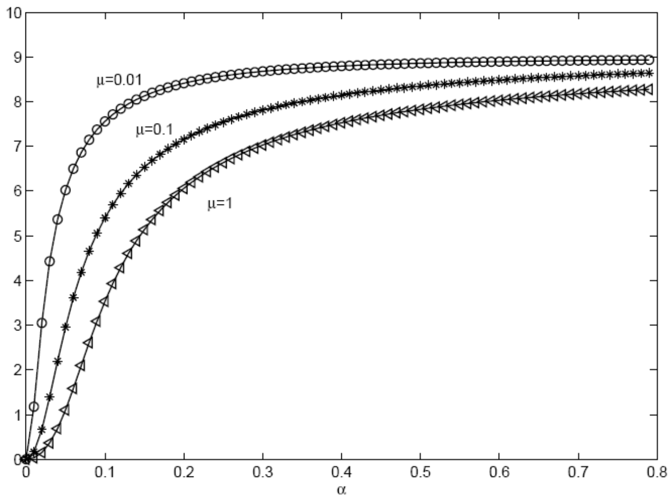
Then we can get the formula for the n^{th} moment of the waiting time W by differentiating the above relation with respect to s at the point $s = 0$. This completes the proof.

Remark 1. A recursive scheme for computing any arbitrary moment of the waiting time process can be derived similarly to that in Falin and Artalejo (1998) (Appendix B) and thus omitted here.

6 Numerical illustrations

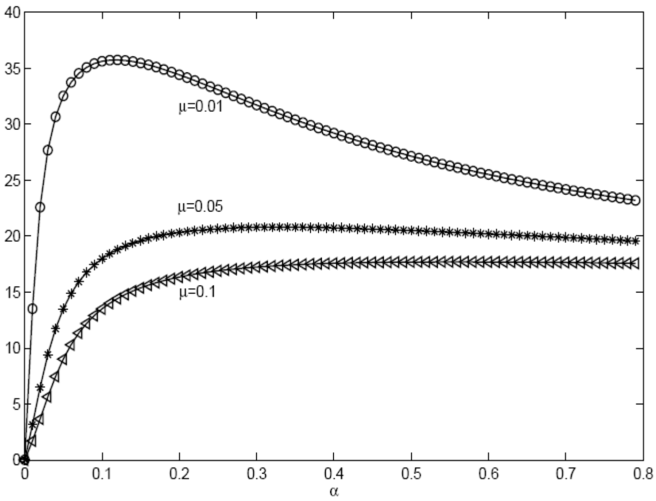
In this section, we present some numerical examples to study the effect of the parameters on the main system performance. To this end, we consider the model with $K = 10$ sources and assume that the service times of two phase service are exponentially distributed with mean $\beta_{11} = \beta_{21} = 1$.

Figure 2 Mean value of retrial sources E[N] versus α ($p = 0.5$)



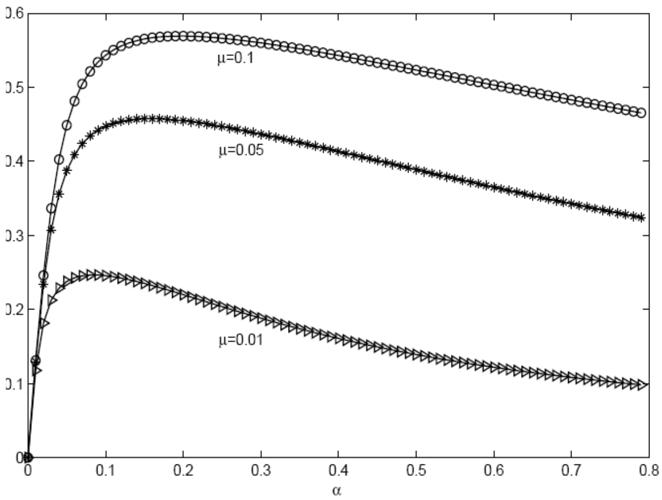
The effect of the rate of generating of primary calls, α, on the mean number of retrial sources and the mean waiting time is showed in Figures 2–3. Figure 2 shows how the mean queue length $E[N]$ changes with the rate of generating of primary calls, α, for different retrial rate μ. It can be observed that when the retrial rate μ takes the same value, $E[N]$ will increase as α increases. It is easy to see that if the arrival rate of primary customers increases, more sources are more likely to find that the server is unavailable upon arrival and then enter the retrial orbit which increases the queue length. However, for a given value α, $E[N]$ will decrease as retrial rate μ increases. That is, more repeated requests will be served so that the number of retrial sources decrease.

Figure 3 Mean waiting time E[W] versus α (p = 0.5)



Similar observation holds for Figure 3. However, in the situation with μ = 0.01 the expected waiting time first increases to an maximum value and after that it decreases as α increases. In both cases, we assume that $1 - p = 0.5$.

Figure 4 Blocking probability P_B versus α (p = 0.5)



The effect of the rate of generating of primary calls, α, on the blocking probability is showed in Figure 4. When the rate of generating of primary calls α takes the same value, the blocking probability P_B is increasing as a function of the retrial rate μ, as we expected. As μ increases, repeated requests in the retrial orbit have more chances to be served and as a result, the primary sources who find the server occupied will increase the blocking probability of the system and the blocked requests have to wait in the retrial orbit.

Figure 5 Mean waiting time E[W] versus 1 – p (μ = 0.01)

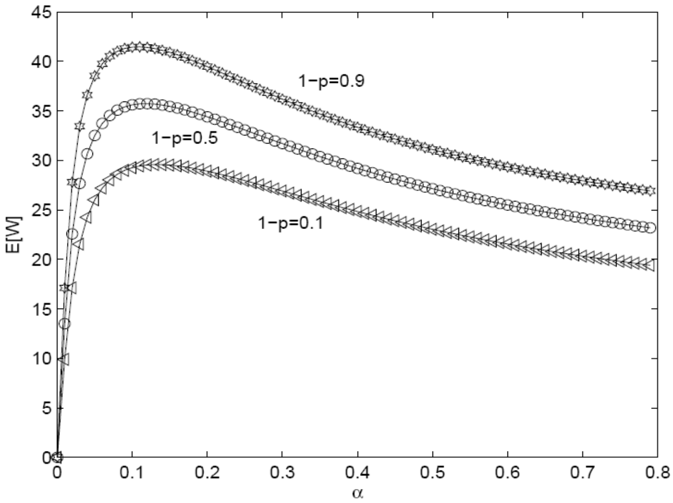


Figure 5 depicts a picture of the effect of the second optional probability, $1-p$, and the rate of generating of primary calls, α, on the mean waiting time. In this case, we have chosen μ = 0.01. As is to be expected, with the increasing of $1-p$, the number of customers in the system decreases and then $E(W)$ increases which agrees with one's intuition.

7 Conclusions

In this paper we have carried out the performance analysis of a single server retrial queueing system with finite sources and two-phase service. All customers demand the FPS, whereas only some of them demand the SPS. We derived some important results on the length of the queue, busy period and waiting time, which are very helpful to provide managerial insight. Furthermore, there are still some research directions for future work. For example, one may take some practical factors such as server breakdowns, vacations and disasters into account, which have wide potential applications in communication networks and computer networks.

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