An Algorithmic Approach for the Analysis of Finite-Source M/GI/1 Retrial Queueing Systems with Collisions and Server Subject to Breakdowns and Repairs

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Abstract. In this paper retrial queuing systems with a finite number of sources and collisions of the customers is considered, where the server is subjects to random breakdowns and repairs depending on whether it is idle or busy. The novelty of this system comparing to the previous ones is that the service time is assumed to follow a general distribution while the source times, retrial times, servers lifetime and repair time are supposed to be exponentially distributed. A new numerical algorithm for finding the joint probability distribution of the number of customers in the system and the server's state is proposed. Several numerical examples and Figures show the effect of different input parameters on the main steady state performance measures, such as mean response and waiting time of the customers, probability of collision and retrials.

Keywords: Finite-source queuing system · Closed queuing systems · Retrial queue · Collision · Server breakdowns and repairs · Unreliable server · Asymptotic analysis · Method of residual service time · Method of elapsed service time

1 Introduction

Finite-source retrial queues are very useful and effective stochastic systems to model several problems arising in telephone switching systems, telecommunication networks, computer networks and computer systems, call centers, wireless communication systems, etc. To see their importance the interested reader is referred to the following works and references cited in them, for example [3,5–7,10]. Searching the scientific databases we have noticed that relatively just a small number of papers have been devoted to systems when the arriving calls (primary or secondary) causes collisions to the request under service and both

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go to the orbit, see for example [1,9,11,21]. It should be noted that collisions decreases the effectiveness of the system performance and that is why new protocols should be developed to avoid the collision but unfortunately it cannot be neglected, see [4,8,16]. This fact shows the importance of the mathematical modeling of such systems.

Nazarov and his research group developed a very effective asymptotic method [20] by the help of which various systems have been investigated. Concerning to finite-source retrial systems with collision we should mention the following papers [12–15,18].

Sztrik and his research group have been dealing with systems with unreliable server/s as can be seen, for example in [2,23] and that is why it was understandable that the two research groups started cooperation in 2017.

The primary aim of the present paper is to give a new numerical algorithm for finding the joint probability distribution of the number of customers in the system and the server's state. The method of supplementary variable is used by introducing the residual service time to derive the system of steady state Kolmogorov equations. An effective algorithmic approach is proposed to get the solution of these equations resulting the steady state distribution of the underlying process. Several numerical examples and Figures show the effect of different input parameters on the main steady state performance measures, such as mean response and waiting time of the customers, probability of collision and retrials. The present model is a generalization of the $M/G/1//N$ retrial system treated in [14] where the server was reliable and the $M/M/1//N$ system with unreliable server analyzed in [17].

The rest of the paper is organized as follows. In Sect. 2 description of the model is given, the corresponding multi-dimensional non-Markov process is defined. Then in Sect. 3 by the help of the residual service time technique the corresponding steady state Kolmogorov equations are derived. Section 4 is devoted to the solution of these equations by proposing and new algorithmic approach and important performance measures are defined. In Sect. 5 several numerical examples and Figures show the effect of different input parameters on the main steady state performance measures and some comments are made. Finally, the paper ends with a Conclusion and some future plans are highlighted.

2 Model Description and Notation

Let us consider a closed retrial queuing system of type $M/GI/1//N$ with collision of the customers and an unreliable server. The number of sources is N and each of them can generate a primary request with rate λ/N. A source cannot generate a new call until the end of the successful service of this customer. If a primary customer finds the server idle and not failed, he enters into service immediately, in which the required service time has a probability distribution function $B(x)$. Let us denote its service rate function by $\mu(y) = B^{'}(y)(1 - B(y))^{-1}$ and its Laplace -Stieltjes transform by $B^{*}(y)$, respectively. Otherwise, if the server is busy, an arriving (primary or repeated) customer involves into collision with the customer under service and they both moves into the orbit. The retrial times of requests are exponentially distributed with rate σ/N. We assume that the server is unreliable, that is its lifetime is supposed to be exponentially distributed with failure rate γ_0 if the server is idle and with rate γ_1 if it is busy. When the server breaks down, it is immediately sent for repair and the recovery time is assumed to be exponentially distributed with rate γ_2. We deal with the case when the server is down all sources continue generation of customers and send it to the server, similarly customers may retry from the orbit to the server but all arriving customers immediately go into the orbit. Furthermore, in this unreliable model we suppose that the interrupted request goes to the orbit immediately and its next service is independent of the interrupted one. The explanation of using λ/N, and σ/N is that in a consecutive paper we would like to investigate the same system by means of asymptotic methods as N tends to infinity and we would like to compare the asymptotic results to the exact ones. All random variables involved in the model construction are assumed to be independent of each other. Let $Q(t)$ be the number of customers in the system at time t, that is, the total number of customers in the orbit and in service. Similarly, let $C(t)$ be the server's state at time t, that is

$$C(t) = \begin{cases} 0, & \text{if the server is idle,} \\ 1, & \text{if the server is busy,} \\ 2, & \text{if the server is down} \\ & \text{(under repair).} \end{cases}$$

Thus, we will investigate the process $\{C(t), Q(t)\}$, which is not a Markov-type unless the required service time is exponentially distributed. Using the supplementary variable method let us introduce the random process $Z(t)$, equal to the residual service time, that is the time interval from moment t until the end of the successful service. It should be noted that the other standard method is to introduce the elapsed service time as the continuous component, see for example [6,25,26] where the resulting Kolmogorov equations are solved by the help of so-called discrete transform. This approach is more common but in our case the residual service time method is more effective as we will show it later on.

As we can see $\{C(t), Q(t), Z(t)\}$ is a three-dimensional Markov process, which has variable number of components, depending on the server's state, since the component $Z(t)$ is determined only in those moments when the server is busy, that is $C(t) = 1$.

3 Kolmogorov Equations for the Probability Distribution

Let us define the following probabilities

$$\begin{aligned} &P_k(j,t) = P\{C(t) = k, Q(t) = j\}, \qquad k = 0, 2 \\ &P_1(j,z,t) = P\{C(t) = 1, Q(t) = j, Z(t) < z\}. \end{aligned}$$

Since the introduction of the residual service time is not so standard as the elapsed service time approach we derive the Kolmogorov equations in more details, namely we can write

$$P_0(0,t+\Delta t) = P_0(0,t)(1-\lambda\Delta t)(1-\gamma_0\Delta t) + \gamma_2\Delta t P_2(0,t) + P_1(1,\Delta t,t) + o(\Delta t), \tag{1}$$

$$\begin{aligned} P_1(1,z-\Delta t,t+\Delta t) =& \Big[P_1(1,z,t) - P_1(1,\Delta t,t)\Big]\Big(1-\lambda\frac{N-1}{N}\Delta t\Big)(1-\gamma_1\Delta t) \\ &+ P_0(0,t)\lambda B(z)\Delta t + P_0(1,t)\frac{\sigma}{N}B(z)\Delta t + o(\Delta t), \end{aligned} \tag{2}$$

$$P_2(0,t+\Delta t) = P_2(0,t)(1-\lambda\Delta t)(1-\gamma_2\Delta t) + \gamma_0\Delta t P_0(0,t) + o(\Delta t), \tag{3}$$

$$\begin{aligned} P_0(j,t+\Delta t) =& P_0(j,t)(1-\lambda\frac{N-j}{N}\Delta t)(1-\gamma_0\Delta t)\Big(1-\frac{j}{N}\sigma\Delta t\Big) + P_1(j+1,\Delta t,t) \\ &+ P_1(j-1,t)\lambda\frac{N-j+1}{N}\Delta t + P_1(j,t)\frac{(j-1)\sigma}{N}\Delta t + P_2(j,t)\gamma_2\Delta t + o(\Delta t), \end{aligned} \tag{4}$$

$$\begin{aligned} &P_1(j,z-\Delta t,t+\Delta t) = \\ &\Big[P_1(j,z,t) - P_1(j,\Delta t,t)\Big]\Big(1-\lambda\frac{N-j}{N}\Delta t\Big)(1-\gamma_1\Delta t)\Big(1-\frac{j-1}{N}\sigma\Delta t\Big) \\ &+ P_0(j-1,t)\lambda\frac{N-j+1}{N}B(z)\Delta t + P_0(j,t)\frac{j\sigma}{N}B(z)\Delta t + o(\Delta t), \end{aligned} \tag{5}$$

$$\begin{aligned} P_2(j,t+\Delta t) =& P_2(j,t)\Big(1-\lambda\frac{N-j}{N}\Delta t\Big)(1-\gamma_2\Delta t) + \gamma_0\Delta t P_0(j,t) \\ &+ P_2(j-1,t)\lambda\frac{N-j+1}{N}\Delta t + P_1(j,t)\gamma_1\Delta t + o(\Delta t) \end{aligned} \tag{6}$$

Assuming that system is operating in steady state, then from the above relations it is not difficult to get the system of equations for the stationary probability distribution $P_0(j)$,$P_1(j,z)$,$P_2(j)$, $\quad j = 0,...,N$ in a shorter form, namely we have

$$\begin{aligned} &-\left[\lambda\frac{N-j}{N} + \sigma\frac{j}{N} + \gamma_0\right]P_0(j) + \frac{\partial P_1(j+1,0)}{\partial z} + \lambda\frac{N-j+1}{N}P_1(j-1) \\ &\qquad + \frac{j-1}{N}\sigma P_1(j) + \gamma_2 P_2(j) = 0\ , \\ &\frac{\partial P_1(j,z)}{\partial z} - \frac{\partial P_1(j,0)}{\partial z} - \left[\lambda\frac{N-j}{N} + \sigma\frac{j-1}{N} + \gamma_1\right]P_1(j,z) \\ &\qquad + \lambda\frac{N-j+1}{N}B(z)P_0(j-1) + \frac{j}{N}\sigma B(z)P_0(j) = 0\ , \\ &-\left[\lambda\frac{N-j}{N} + \gamma_2\right]P_2(j) + \lambda\frac{N-j+1}{N}P_2(j-1) + \gamma_0 P_0(j) + \gamma_1 P_1(j) = 0\ . \end{aligned} \tag{7}$$

where the meaningless probabilities are zero.

4 Numerical Algorithm for Finding the Probability Distribution of the System State and Performance Measures

4.1 Algorithmic Approach for the Steady State Distribution

In order to find the probability distribution of the number of customers in the system, we will solve system (7) numerically. We first obtain some very important equalities used later on.

Let us consider the second equation of system (7) for case $j = 1$, that is

$$\frac{\partial P_1(1,z)}{\partial z} - \frac{\partial P_1(1,0)}{\partial z} - \left[\lambda \frac{N-1}{N} + \gamma_1\right] P_1(1,z) \\ + \lambda B(z) P_0(0) + \frac{\sigma}{N} B(z) P_0(1) = 0. \tag{8}$$

The solution of this equation can be written in the form

$$P_1(1,z) = e^{\left[\lambda \frac{N-1}{N} + \gamma_1\right] z} \int\limits_0^z e^{-\left[\lambda \frac{N-1}{N} + \gamma_1\right] y} \left\{ \frac{\partial P_1(1,0)}{\partial z} \right. \\ \left. - \left[\lambda P_0(0) + \frac{\sigma}{N} P_0(1) \right] B(y) \right\} dy. \tag{9}$$

Then by carrying out the limiting transition at $z \to \infty$ we obtain that the first factor of the right part of equality (9) in a limiting condition tends to infinity, therefore we can conclude that the second factor will be equal to zero, that is

$$\int\limits_0^\infty e^{-\left[\lambda \frac{N-1}{N} + \gamma_1\right] y} \left\{ \frac{\partial P_1(1,0)}{\partial z} - \left[\lambda P_0(0) + \frac{\sigma}{N} P_0(1) \right] B(y) \right\} dy = 0,$$

from which it is not difficult to obtain that

$$\frac{\partial P_1(1,0)}{\partial z} = \left[\lambda P_0(0) + \frac{\sigma}{N} P_0(1)\right] B^* \left(\lambda \frac{N-1}{N} + \gamma_1\right). \tag{10}$$

We can perform similar transformations for the second equation of system (7) for the general case and, as a result we obtain

$$\frac{\partial P_1(j,0)}{\partial z} = \left[\lambda \frac{N-j+1}{N} P_0(j-1) + \frac{j}{N} \sigma P_0(j)\right] B^* \left(\lambda \frac{N-j}{N} + \frac{j-1}{N} \sigma + \gamma_1\right). \tag{11}$$

In Eq. (8) let us execute the limiting transition at $z \to \infty$ then we get

$$\frac{\partial P_1(1,0)}{\partial z} \left[\lambda \frac{N-1}{N} + \gamma_1\right] P_1(1) = \lambda P_0(0) + \frac{\sigma}{N} P_0(1). \tag{12}$$

Similarly, for the general case, that is for j, using the second equation of system (7), we can obtain

$$\frac{\partial P_1(j,0)}{\partial z}\left[\lambda\frac{N-j}{N}+\frac{j-1}{N}\sigma+\gamma_1\right]P_1(j)=\lambda\frac{N-j+1}{N}P_0(j-1)+\frac{j}{N}\sigma P_0(j). \tag{13}$$

Let us write down the system of Eq. (7) for the case $j=0$ then we get

$$\begin{aligned}\frac{\partial P_1(1,0)}{\partial z}&=\left[\lambda+\gamma_0\right]P_0(0)-\gamma_2P_2(0),\\ -\left[\lambda+\gamma_2\right]P_2(0)&+\gamma_0P_0(0)=0.\end{aligned} \tag{14}$$

Hence combining equations of the system (7) for case $j=1$ by using Eqs. (10) and (12) we obtain

$$\begin{aligned}&\frac{\partial P_1(1,0)}{\partial z}=\left[\lambda P_0(0)+\frac{\sigma}{N}P_0(1)\right]B^*\left(\lambda\frac{N-1}{N}+\gamma_1\right),\\ &\frac{\partial P_1(1,0)}{\partial z}+\left[\lambda\frac{N-1}{N}+\gamma_1\right]P_1(1)=\lambda P_0(0)+\frac{\sigma}{N}P_0(1),\\ &\left[\lambda\frac{N-1}{N}+\gamma_2\right]P_2(1)=\gamma_0P_0(1)+\gamma_1P_1(1)+\lambda P_2(0),\\ &\frac{\partial P_1(2,0)}{\partial z}=\left[\lambda\frac{N-1}{N}+\gamma_0+\frac{\sigma}{N}\right]P_0(1)-\gamma_2P_2(1)-\lambda P_1(0).\end{aligned} \tag{15}$$

Similarly, using the equations of system (7) and the equalities (11), (13) obtained earlier we can write down the extended system of equations for $2\le j\le N$ as follows

$$\begin{aligned}&\left[\lambda\frac{N-j}{N}+\sigma\frac{j}{N}+\gamma_0\right]P_0(j)=\frac{\partial P_1(j+1,0)}{\partial z}+\lambda\frac{N-j+1}{N}P_1(j-1)\\ &\qquad\qquad+\frac{j-1}{N}\sigma P_1(j)+\gamma_2P_2(j)\ ,\\ &\left[\lambda\frac{N-j}{N}+\gamma_2\right]P_2(j)=\lambda\frac{N-j+1}{N}P_2(j-1)+\gamma_0P_0(j)+\gamma_1P_1(j)\ ,\\ &\frac{\partial P_1(j,0)}{\partial z}=\left[\lambda\frac{N-j+1}{N}P_0(j-1)+\frac{j}{N}\sigma P_0(j)\right]B^*\left(\lambda\frac{N-j}{N}+\frac{j-1}{N}\sigma+\gamma_1\right)\ ,\\ &\frac{\partial P_1(j,0)}{\partial z}+\left[\lambda\frac{N-j}{N}+\frac{j-1}{N}\sigma+\gamma_1\right]P_1(j)=\lambda\frac{N-j+1}{N}P_0(j-1)++\frac{j}{N}\sigma P_0(j)\ .\end{aligned} \tag{16}$$

The joint stationary probability distribution $\Pi_k(j)$ of the server's state and the number of customers in the system is the normalized solution of systems (14)–(16). Starting with $P_0(0) = 1$ using the algorithmic steps after normalization we obtain the probability distribution $\Pi_k(j)$. Thus our proposed algorithmic solution consists of the following steps

1. Put $P_0(0) = 1$.
2. From the second equation of system (14) we get

$$P_2(0) = \frac{\gamma_0}{\lambda + \gamma_2} P_0(0).$$

3. From the first equations of systems (14), (15) we obtain

$$P_0(1) = \frac{N}{\sigma B^*\left(\lambda \frac{N-1}{N} + \gamma_1\right)} \{-\gamma_2 P_2(0) +$$

$$+ \left(\lambda \left[1 - B^*\left(\lambda \frac{N-1}{N} + \gamma_1\right)\right] + \gamma_0\right) P_0(0)\right\}.$$

4. From the first equation of system (14) and second equation of system (15) we have

$$P_1(1) = \frac{1}{\lambda \frac{N-1}{N} + \gamma_1} \left\{ \frac{\sigma}{N} P_0(1) - \gamma_0 P_0(0) + \gamma_2 P_2(0) \right\}.$$

5. From the third equation of system (15) we determine

$$P_2(1) = \frac{1}{\lambda \frac{N-1}{N} + \gamma_2} \left\{ \gamma_0 P_0(1) + \gamma_1 P_1(1) + \lambda P_2(0) \right\}.$$

6. For general case, that is for $2 \le j \le N$, from system (16) it is not difficult to obtain formulas for calculating $P_k(j)$ in the form

$$P_0(j) = \frac{1}{j\sigma B^*\left(\lambda \frac{N-j}{N} + \frac{j-1}{N}\sigma + \gamma_1\right)} \Bigg\{ -\lambda(N-j+2)P_1(j-2)$$

$$+ \left(\lambda(N-j+1)\left[1 - B^*\left(\lambda \frac{N-j}{N} + \frac{j-1}{N}\sigma + \gamma_1\right)\right] + (j-1)\sigma\right.$$

$$\left. + \gamma_0 N \right) P_0(j-1) - (j-2)\sigma P_1(j-1) - \gamma_2 N P_2(j-1) \Bigg\},$$

$$P_1(j) = \frac{1}{\lambda(N-j) + \sigma(j-1) + \gamma_1 N} \{ -\left[\sigma(j-1) + \gamma_0 N\right] P_0(j-1)$$

$$+ j\sigma P_0(j) + \lambda(N-j+2)P_1(j-2) + (j-2)\sigma P_1(j-1) + \gamma_2 N P_2(j-1) \},$$

$$P_2(j) = \frac{1}{\lambda(N-j) + \gamma_2 N} \left[\gamma_0 N P_0(j) + \gamma_1 N P_1(j) + \lambda(N-j+1)P_2(j-1) \right],$$

$P_1(0) = 0$ by convention.

7. The solution obtained in the previous steps does not satisfy the normalization condition. For the normalizing constant let us calculate the sum

$$d = \sum_{j=0}^{N} \left[P_0(j) + P_1(j) + P_2(j) \right],$$

where $P_k(j)$ is the quantities obtained in the previous steps.
8. To calculate the two-dimensional probability distribution $\Pi_k(j)$ carry on the normalization, that is

$$\Pi_k(j) = \frac{1}{d} P_k(j), \quad k = 0, 1, 2, \quad j = 0, 1, ..., N.$$

9. The marginal distribution of the number of customers in the system $\Pi(j)$, and the server's state Π_k, respectively can be calculated as follows

$$\Pi(j) = \Pi_0(j) + \Pi_1(j) + \Pi_2(j), j = 0, ..., N, \qquad \Pi_k = \sum_{j=0}^{N} \Pi_k(j), k = 0, 1, 2.$$

4.2 Performance Measures

To show the effect of the input parameters on the operation of the system let us define the most important characteristics which can be determine directly from the steady state probabilities. Unfortunately only mean values are obtained but our intention is to continue the research to get the distribution of the response and waiting time of the customers, distribution of the number of retrials just to mention some.

– *Mean number of customers in the system* $\overline{Q}$

$$\overline{Q} = \sum_{j=0}^{N} j \Pi(j), \tag{17}$$

– *Mean arrival rate* $\overline{\lambda}$

$$\overline{\lambda} = (N - \overline{Q}) \frac{\lambda}{N}, \tag{18}$$

– *Mean response time* $\overline{T}$ can be obtained by the Little-formula

$$\overline{T} = \frac{\overline{Q}}{\overline{\lambda}}, \tag{19}$$

– *Mean number of customers in the orbit* $\overline{O}$

$$\overline{O} = \overline{Q} - \Pi_1, \tag{20}$$

– *Mean waiting time in the orbit* $\overline{W}$

$$\overline{W} = \frac{\overline{O}}{\overline{\overline{\lambda}}}, \tag{21}$$

– *Mean total service time* $E(T_S)$

$$E(T_S) = \overline{T} - \overline{W}, \tag{22}$$

– *Probability of collision of a customer arriving from the source (Primary Customer)* P_{PC}

$$P_{PC} = \frac{\sum_{k=1}^{N}(N-k)\frac{\lambda}{N}\varPi_1(k)}{\sum_{j=0}^{N}(N-j)\frac{\lambda}{N}(\varPi_0(j)+\varPi_1(j))}, \tag{23}$$

– *Probability of collision of a customer arriving from the orbit (Secondary Customer)* P_{SC}

$$P_{SC} = \frac{\sum_{j=1}^{N}(j-1)\frac{\sigma}{N}\varPi_1(j)}{\sum_{j=0}^{N} j\frac{\sigma}{N}\varPi_0(j) + \sum_{j=1}^{N}(j-1)\frac{\sigma}{N}\varPi_1(j)}, \tag{24}$$

– *Probability of collision* P_C

$$P_C = \tag{25}$$
$$\frac{\sum_{j=1}^{N}\left[(N-j)\frac{\lambda}{N}+(j-1)\frac{\sigma}{N}\right]\varPi_1(j)}{\sum_{j=0}^{N}\left[(N-j)\frac{\lambda}{N}+j\frac{\sigma}{N}\right]\varPi_0(j)+\sum_{j=0}^{N}(N-j)\frac{\lambda}{N}\varPi_1(j)+\sum_{j=1}^{N}(j-1)\frac{\sigma}{N}\varPi_1(j)},$$

– *Probability of retrial* P_R

$$P_R = \tag{26}$$
$$\frac{\sum_{j=0}^{N}(N-j)\frac{\lambda}{N}\Big(\varPi_1(j)+\varPi_2(j)\Big)+\sum_{j=1}^{N}(j-1)\frac{\sigma}{N}\varPi_1(j)+\sum_{j=1}^{N} j\frac{\sigma}{N}\varPi_2(j)+\gamma_1\varPi_1}{\sum_{j=0}^{N}(N-j)\frac{\lambda}{N}\varPi(j)+\sum_{j=1}^{N}(j-1)\frac{\sigma}{N}\varPi_1(j)+\sum_{j=1}^{N} j\frac{\sigma}{N}\Big(\varPi_0(j)+\varPi_2(j)\Big)+\gamma_1\varPi_1}.$$

5 Numerical Examples

The proposed algorithm has been tested by the numerical results of [13] where the service time is exponentially distributed and the server is reliable, [14] in which the service time is generally distributed and the server is reliable, and [19] where the service time is exponentially distributed and the server is subject to breakdowns and repairs. Finally, in [24] the present model has been analyzed by means of stochastic simulation.

Table 1. Numerical values of model parameters

Case studies								
No.	N	λ	σ	α	β	γ_0	γ_1	γ_2
Fig. 1	30	2	10	0.5	0.5	0.1	0.2	1
Fig. 2	30	2	10	0.5, 1, 2	0.5, 1, 2	0.1	0.2	1
Fig. 3	100	0.01..10	1	0.5, 1, 2	0.5, 1, 2	0.1	0.1	1
Fig. 4	100	0.01..0.2	1	0.5, 1, 2	0.5, 1, 2	0.1	0.1	1
Fig. 5	100	0.01..0.2	1	0.5, 1, 2	0.5, 1, 2	0.1	0.1	1
Fig. 6	100	0.01..0.25	0.1	0.5	0.5	0.5	0.5	1
Fig. 7	100	0.01..0.15	0.1	1	1	0.5	0.5	1
Fig. 8	100	1	1	1	1	0.05..1	0.05..1	2

In our examples we will choose gamma distributed service time S with a shape parameter α and scale parameter β, with Laplace-Stieltjes transform $B^*(\delta)$ of the form

$$B^*(\delta) = \left(1 + \frac{\delta}{\beta}\right)^{-\alpha},$$

in the case when $\alpha = \beta$, that is when the average service time will be equal to unit.

It can be shown that

$$\mathsf{E}(S) = \frac{\alpha}{\beta}, \quad Var(S) = \frac{\alpha}{\beta^2}, \quad V_S^2 = \frac{1}{\alpha},$$

where V_S^2 denotes the squared coefficient of variation of S. This distribution allows us to show the effect of the distribution on the main performance measures, because dealing with the same mean we can see the impact of the variance, too.

From the system probabilities the well known system characteristics are calculated. The most interesting performance characteristics obtained by these tools are graphically presented in this section. On the Figures the lines represent different working assumptions or cases (e.g. different parameters of the distribution of the service time). The input parameters are listed in Table 1.

Figures 1 and 2 display distributions of the steady-state system probabilities where values of x-axes represent the numbers of customers staying in the system, i.e. the states of the system. On the other Figures the effects of a running parameter are shown. In Table 1 a parameter running from n to m is denoted by $n..m$. If the effect of an other parameter is also considered, a separate curve is presented for each values of that parameter, and these values are listed in Table 1, as well.

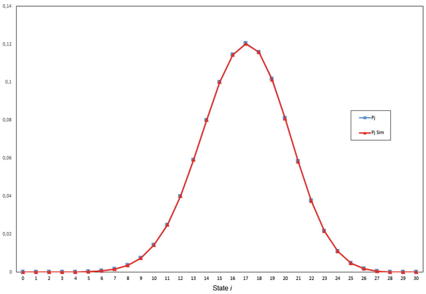

Fig. 1. Comparison of numerical and simulation results

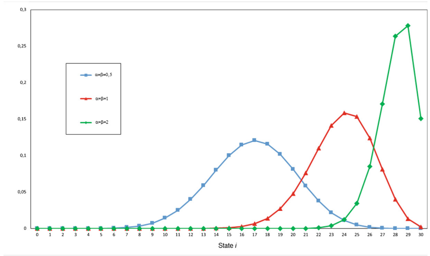

Fig. 2. Comparison the distributions for different α and β parameters

On Fig. 1 the numerical and simulation results for the steady-state probabilities are compared to each other. As we can see the values are very close to each other, so the two curves are identical illustrating that the numerical and simulation procedures operate correctly.

Figure 2 shows the effect of the different values of the shape α, and scale/rate β parameters. The curves represent the cases of $\alpha = \beta$ with values $0.5, 1, 2,$ respectively. Thus, the expected values of the service times are equal but the variances are different. For higher values of α, and β parameters, the standard deviation and the coefficient of variance will be smaller. For small values of parameters, i.e. high value of standard deviation, the distribution is more tailed than for higher values of α, and β.

On Fig. 3 the mean waiting time can be seen in different cases. For Case 1, 2, and 3 the values of α, and β are $0.5, 1, 2$, respectively ($\alpha = \beta$ for all cases). A maximum point can be observed for this performance measure, as the arrival rate increases. In retrial systems this maximum feature is an unexpected and quite unique phenomenon. Many times there exists a combination of parameters, for which the response time, waiting time or queue length have a maximum point, see for example in [5,6,22,25].

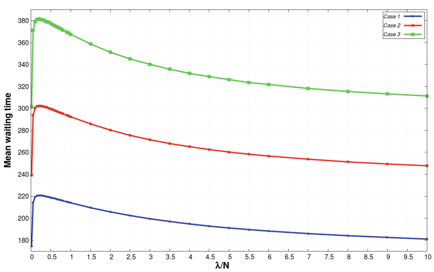

Fig. 3. Mean waiting time $\overline{W}$ vs arrival intensity from the source

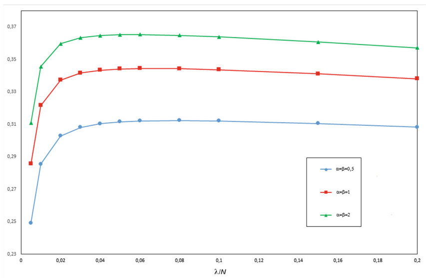

Fig. 4. Probability of collision for primary customers P_{PC} vs arrival intensity from the source

The mentioned maximum feature can be observed on Fig. 4, as well. Here again the arrival rate is the running parameter. The different lines correspond to the different α, and β parameters as on Fig. 2 and 3. As mentioned above, this maximum point can be achieved only a specific set of parameters. With the parameters of Fig. 4, the probabilities of retrial are computed and displayed on Fig. 5. Here there are no maximum points for the probabilities. The P_R values increase with the increasing arrival rate. But when some parameters (retrial rate, failure rates) are modified, the following results can be obtained: for $\alpha = \beta = 0.5$ the Fig. 6, and for $\alpha = \beta = 1$ the Fig. 7. The maximum feature and the decreasing trend of the probabilities can be seen on both Figures. A similar Figure could be generated for $\alpha = \beta = 2$ case, too.

Finally, Fig. 8 displays the result of the effect of modification of failure rates. γ_0, and γ_1 are modified parallel, the same way, so for each point $\gamma_0 = \gamma_1$. The range of the parameters can be found in Table 1. The P_{PC}, P_C, and P_R probabilities are displayed, but only two lines are in the Figure. The values of P_{PC}, and P_C are so close (not identical) to each other, that only one line can be seen for these two parameters. The results show what is expected, that is as the failure rate increasing more and more requests are sent to the orbit causing retrials, but the chance of collision is decreasing since the server is broken.

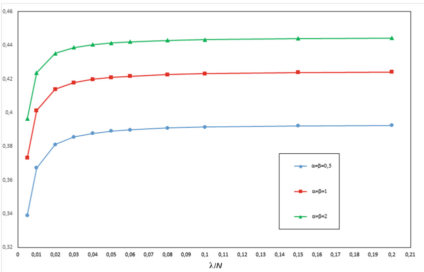

Fig. 5. Probability of retrial P_R vs arrival intensity from the source

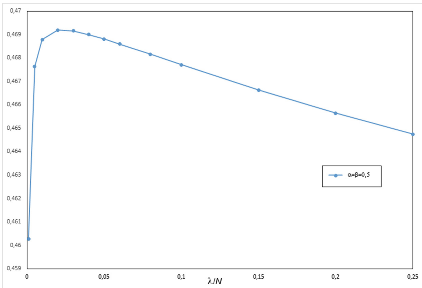

Fig. 6. Probability of retrial P_R vs arrival intensity from the source

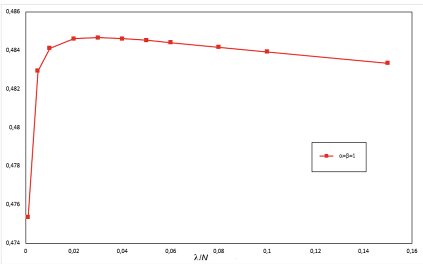

Fig. 7. Probability of retrial P_R vs arrival intensity from the source

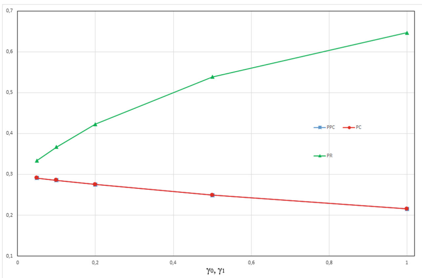

Fig. 8. Values of probabilities P_{PC}, P_C, and P_R vs failure rate

6 Conclusion

In this paper finite source $M/GI/1$ retrial queuing systems with collisions of the customers and an unreliable server were considered. Applying the method of residual service times as supplementary variable the steady state Kolmogorov equations were solved by means of a new algorithmic approach. The main performance measures were defined and several numerical sample examples illustrated the effect of the input parameters on these characteristics. In the near future, for the considered system we plan to investigate the distribution of the number of transitions of the customer into the orbit, distribution of the sojourn time of the customer in the system and other system performance descriptors.

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