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PROBABILISTIC ANALYSIS OF A REDUNDANT REPAIRABLE SYSTEM WITH TWO SERVICE OPERATIONS

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A redundant system with two types of maintenance services, with single repair server and single replacement server, is considered. The replacement and repair times are random variables. Mathematical model for dependability and performance analysis is constructed and investigated in transient (time-dependent mode). An explicit solution in terms of matrix exponential calculus for transient probabilities is obtained.

1. Introduction

In the traditional mathematical theory of reliability, namely in redundancy theory, the replacement of the failed unit by redundant one was quite convincingly regarded as instantaneous, and the duration of repair of the failed unit significantly small compared to its lifetime. In most studies of reliability of the redundant systems, the replacement of the failed main unit by the redundant one is not considered as a separate, independent maintenance operation. Under these conditions, the mean duration of downtime is indeed negligible.

As a matter of fact, nowadays it is important to estimate not only the mean downtime, but also the nature of its variation around mean value. The problem of a full probabilistic analysis of downtime (including replacement time) is very urgent. A natural necessity arose to construct and investigate maintenance models which consider two types of maintenance operations: 1) replacement of the failed main unit by the redundant one, and 2) repair of the failed unit (both main and redundant).

Simultaneously, duplex and triplex systems are most widespread in redundancy practice. In the presented paper we investigate one of the interesting triplex systems. Particularly a redundant system with two types of maintenance services, with single repair server and single replacement server, is considered. The replacement and repair times are random variables. A mathematical model for dependability and performance analysis is constructed and investigated. An explicit solution for transient probabilities in terms of matrix exponential is obtained. The result of the research can be used for further analysis of the system to derive some important characteristics with different distribution functions of renewal time.

Moreover, in classical reliability theory, as a rule, mathematical models of the considered systems, including semi-Markov models, have been studied in stationary (steady-state) mode. However, it is interesting to investigate the process on an initial stage, when state probabilities still significantly depend on time. Firstly, such kind of research allows us to establish how quickly the probabilities aspire to the final values and, hence, to define then with sufficient accuracy, after which it is possible to consider a stationary process. Secondly, in applications of the mathematic theory of reliability, the behavior of a semi-Markov process is of great interest only at the initial stage, when the process does not advance ito the stationary regime yet.

In the presented paper, this kind of analysis is done.

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2. Description of the system and its mathematical model

The redundant technical system consists of two main units and one redundant unit. The failure of the main unit happens with intensity α and that of the redundant one with intensity β, $0 < \beta \leqslant \alpha$. The failed main unit is replaced by an operating standby element after the replacement of the latter. Meanwhile, the failed elements, both main and standby, are transmitted for renewal. We have one server performing renewal and one server for replacement. Suppose that the replacement time of a failed element is a random variable with exponential distribution of rate μ. The renewal time distribution function G is arbitrary. The replaced unit renews all of its initial qualities and is included in the group of redundant units. Denote the renewal rate by $\eta(u)$ so that $\eta(u) = \frac{g(u)}{1-G(u)}$, where $g(u) = G'(u)$.

To define the states of the considered system at the time instant t, introduce the random processes

- $i(t)$ is the number of units missing in the group of main units;
- $j(t)$ is the number of non-operative (failed) units in the system;
- $\xi(t)$ is the time interval length from the beginning of the renewal operation to the time instant t.

In order to describe the system state, we define the following probability characteristics:

$$P(i,t) = P\{i(t) = i,\ \ j(t) = 0\},\ \ i = 0,1,2,$$

$$q(i,j,t,u) = \lim_{h\to 0}\Big(\frac{1}{h}\,P\{i(i) = i,\ \ j(t) = j,\ \ u < \xi(t) < u+h\}\Big),\ \ i = 0,1,2,\ \ j = 1,\dots,1+i.$$

The functions $P(j,t)$ and $q(i,j,t,u)$ have great theoretical and practical values, as they allow us to easily express other characteristics of the system.

Theorem 1. *Let the functions* $P(i,t)$, $i = 0,1,2$, *have continuous derivatives when* $t > 0$; *then they satisfy the following system of integral-differential equations*:

$$\begin{aligned}
\frac{dP(0,t)}{dt} &= -(2\alpha+\beta)P(0,t) + \mu P(1,t) + \int_0^t q(0,1,t,u)\eta(u)\,du,\\
\frac{dP(1,t)}{dt} &= -(\alpha+\beta+\mu)P(1,t) + \mu P(2,t) + \int_0^t q(1,1,t,u)\eta(u)\,du,\\
\frac{dP(2,t)}{dt} &= -(2\beta+\mu)P(2,t) + \int_0^t q(2,1,t,u)\eta(u)\,du
\end{aligned} \tag{1}$$

with the initial conditions $P(0,0) = 1,\ P(1,0) = 0,\ P(2,0) = 0.$

Theorem 2. *Let the functions* $q(i,j,t,u)$, $i = 0,1,2$, $j = 1,\dots,i+1$, *have continuous partial* *derivatives when $t > 0$, $u \geqslant 0$. Then they satisfy the following system of partial differential equations*:

$$\begin{aligned}
\frac{\partial q(0,1,t,u)}{\partial t} + \frac{\partial q(0,1,t,u)}{\partial u} &= -(2\alpha + \eta(u))q(0,1,t,u) + \mu q(1,1,t,u),\\
\frac{\partial q(1,1,t,u)}{\partial t} + \frac{\partial q(1,1,t,u)}{\partial u} &= -(\alpha + \mu + \eta(u))q(1,1,t,u) + \mu q(2,1,t,u),\\
\frac{\partial q(1,2,t,u)}{\partial t} + \frac{\partial q(1,2,t,u)}{\partial u} &= -(\alpha + \eta(u))q(1,2,t,u) + 2\alpha q(0,1,t,u) + \mu q(2,2,t,u),\\
\frac{\partial q(2,1,t,u)}{\partial t} + \frac{\partial q(2,1,t,u)}{\partial u} &= -(\beta + \mu + \eta(u))q(2,1,t,u),\\
\frac{\partial q(2,2,t,u)}{\partial t} + \frac{\partial q(2,2,t,u)}{\partial u} &= -(\mu + \eta(u))q(2,2,t,u) + \alpha q(1,1,t,u) + \beta q(2,1,t,u),\\
\frac{\partial q(2,3,t,u)}{\partial t} + \frac{\partial q(2,3,t,u)}{\partial u} &= -\eta(u)q(2,3,t,u) + \alpha q(1,2,t,u)
\end{aligned} \tag{2}$$

with the boundary conditions

$$\begin{aligned}
q(0,1,t,0) &= \beta P(0,t),\\
q(1,1,t,0) &= 2\alpha P(0,t) + \beta P(1,t) + \int_0^t q(1,2,t,u)\eta(u)\,du,\\
q(1,2,t,0) &= 0,\\
q(2,1,t,0) &= \alpha P(1,t) + 2\beta P(2,t) + \int_0^t q(2,2,t,u)\eta(u)\,du,\\
q(2,2,t,0) &= \int_0^t q(2,3,t,u)\eta(u)du,\\
q(2,3,t,0) &= 0.
\end{aligned}$$

The proof of the theorems is omitted. It can be performed by determining the changes occurring in the states of the system within an infinitesimal time interval.

Consider the part of the system relative to $P(i,t)$.

Denote

$$\int_0^t q(i,1,t,u)\eta(u)\,du := r(i,t).$$

Further, assume that

$$P(t) = \begin{pmatrix} P(0,t)\\ P(2,t)\\ P(2,t)\end{pmatrix}, \quad R(t) = \begin{pmatrix} r(0,t)\\ r(1,t)\\ r(2,t)\end{pmatrix},$$

$$A = \begin{pmatrix} -(2\alpha+\beta) & \mu & 0\\ 0 & -(\alpha+\beta+\mu) & \mu\\ 0 & 0 & -(2\beta+\mu)\end{pmatrix}.$$

In this notation we get the nonhomogeneous linear equation in the vector form

$$\frac{dP(t)}{dt} = AP(t) + R(t), \quad P(0) = \begin{pmatrix}1\\0\\0\end{pmatrix} \tag{3}$$

with the solution

$$P(t) = e^{tA}P(0) + \int_0^t e^{(t-s)A} r(s)\,ds,$$

where e^{tA} is the matrix exponential. We thus see that the solution of differential equation (3) is reduced to finding the matrix exponential.

The characteristic equator for matrix A is given by

$$\begin{vmatrix} -(2\alpha+\beta)-\theta & \mu & 0 \\ 0 & -(\alpha+\beta+\mu)-\theta & \mu \\ 0 & 0 & -(2\beta+\mu)-\theta \end{vmatrix} = 0.$$

Therefore

$$((2\alpha+\beta)+\theta)((\alpha+\beta+\mu)+\theta)((2\beta+\mu)+\theta) = 0.$$

We thus see that $-(2\beta+\mu)$, $-(\alpha+\beta+\mu)$, and $-(2\alpha+\beta)$ are eigenvalues of the matrix and the corresponding eigenvectors $\begin{pmatrix}1 & 0 & 0\end{pmatrix}^\top$, $\begin{pmatrix}\frac{\mu}{\alpha-\mu} & 1 & 0\end{pmatrix}^\top$, $\begin{pmatrix}\frac{\mu^2}{(\alpha-\beta)(2\alpha-\beta-\mu)} & \frac{\mu}{\alpha-\beta} & 1\end{pmatrix}^\top$ are linearly independent.

Assume that

$$C = \begin{pmatrix} 1 & \dfrac{\mu}{\alpha-\mu} & \dfrac{\mu^2}{(\alpha-\beta)(2\alpha-\beta-\mu)} \\ 0 & 1 & \dfrac{\mu}{\alpha-\beta} \\ 0 & 0 & 1 \end{pmatrix}.$$

Since the column vectors are linearly independent, C has an inverse

$$C^{-1} = \begin{pmatrix} 1 & -\dfrac{\mu}{\alpha-\mu} & \dfrac{\mu^2}{(\alpha-\mu)(2\alpha-\beta-\mu)} \\ 0 & 1 & -\dfrac{\mu}{\alpha-\beta} \\ 0 & 0 & 1 \end{pmatrix}$$

and we have

$$A = CJC^{-1},$$

where J is the diagonal matrix whose diagonals are eigenvalues of A

$$J = \begin{pmatrix} -(2\alpha+\beta) & 0 & 0 \\ 0 & -(\alpha+\beta+\mu) & 0 \\ 0 & 0 & -(2\beta+\mu) \end{pmatrix}.$$

Thus $e^{tA} = e^{tCJC^{-1}} = Ce^{tJ}C^{-1}$. Therefore,

$$e^{Jt} = \begin{pmatrix} e^{-(2\alpha+\beta)t} & 0 & 0 \\ 0 & e^{-(\alpha+\beta+\mu)t} & 0 \\ 0 & 0 & e^{-(2\beta+\mu)t} \end{pmatrix} = e^{tTrA}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = e^{tTrA} I,$$

where I is the identity matrix and

$$TrA = -(2\alpha+\beta)t - (\alpha+\beta+\mu)t - (2\beta+\mu) = -(3\alpha+4\beta+2\mu)t.$$

Hence, we have

$$e^{tTrA} = e^{-(3\alpha+4\beta+2\mu)t} \text{ and } e^{Jt} = e^{-(3\alpha+4\beta+2\mu)t} I.$$

It follows that

$$P(t) = e^{tA}P(0) + \int_0^t e^{(t-s)A}R(s)\,ds =$$
$$= e^{tTrA}P(0) + \int_0^t e^{(t-s)TrA}R(s)\,ds = e^{tTrA}P(0) + e^{tTrA}\int_0^t e^{-sTrA}R(s)\,ds$$

and

$$P(t) = e^{-(3\alpha+4\beta+2\mu)t}\Big(P(0) + \int_0^t e^{(3\alpha+4\beta+2\mu)s}R(s)\,ds\Big).$$

So we obtain the expression for $P(i,t)$:

$$\begin{aligned}
P(0,t) &= e^{-(3\alpha+4\beta+2\mu)t}\Big(1 + \int_0^t\int_0^s e^{(3\alpha+4\beta+2\mu)s}q(0,1,s,u)\eta(u)\;du\,ds\Big),\\
P(1,t) &= e^{-(3\alpha+4\beta+2\mu)t}\int_0^t\int_0^s e^{(3\alpha+4\beta+2\mu)s}q(1,1,s,u)\eta(u)\;du\,ds,\\
P(2,t) &= e^{-(3\alpha+4\beta+2\mu)t}\int_0^t\int_0^s e^{(3\alpha+4\beta+2\mu)s}q(2,1,s,u)\eta(u)\;du\,ds.
\end{aligned} \tag{4}$$

The expressions for the integrands in $q(i,1,t,u)$ can be obtained directly from the recurrent system of differential equations (2).

Suppose that $\mu-\alpha \neq 0$, $\alpha-\beta-\mu \neq 0$ and $2\alpha-\beta-\mu \neq 0$. (From the theoretical point of view, these cases are considered independently; in practice these relations are almost impossible.)

From system (2) we obtain

$$\begin{aligned}
q(2,1,t,u) &= (1-G(u))H(2,1,t-u)e^{-(\beta+\mu)u},\\
q(1,1,t,u) &= (1-G(u))\Big(H(1,1,t-u)e^{-(\alpha+\mu)u} + \frac{\mu}{\alpha-\beta}\,H(2,1,t-u)e^{-(\beta+\mu)u}\Big),\\
q(0,1,t,u) &= (1-G(u))\Big(H(0,1,t-u)e^{-2\alpha u} + \frac{\mu}{\alpha-\mu}\,H(1,1,t-u)e^{-(\alpha+\mu)u} +\\
&\quad + \frac{\mu}{2\alpha-\mu-\beta}\,\frac{\mu}{\alpha-\beta}\,H(2,1,t-u)e^{-(\alpha+\mu)u}\Big),\\
q(2,2,t,u) &= (1-G(u))\bigg(H(2,2,t-u)e^{-\mu u} - H(1,1,t-u)e^{-(\alpha+\mu)u} +\\
&\quad + \Big(1 + \frac{\alpha}{\beta}\,\frac{\mu}{\alpha-\beta}\Big)H(2,1,t-u)e^{-(\beta+\mu)u}\bigg),\\
q(1,2,t,u) &= (1-G(u))\bigg(H(1,2,t-u)e^{-\alpha u} - 2H(0,1,t-u)e^{-2\alpha u} +\\
&\quad + \frac{\mu}{\alpha-\mu}\,H(2,2,t-u)e^{-\mu u} + \frac{\mu+\alpha}{\mu-\alpha}\,H(1,1,t-u)e^{-(\alpha+\mu)u} -\\
&\quad - \frac{\mu}{\alpha-\mu-\beta}\,\Big(1 + \frac{\mu}{\alpha-\beta}\,\frac{\alpha}{\beta}\,\frac{\mu+3\beta-2\alpha}{\mu+\beta-2\alpha}\Big)H(2,1,t-u)e^{-(\beta+\mu)u}\bigg),
\end{aligned} \tag{5}$$

$$q(2,3,t,u) = (1-G(u))\Big(H(2,3,t-u) - H(1,2,t-u)e^{-\alpha u} + H(0,1,t-u)e^{-2\alpha u} + \\ + \frac{\alpha}{\alpha-\mu} H(2,2,t-u)e^{-\mu u} + \frac{\alpha}{\alpha-\mu} H(1,1,t-u)e^{-(\alpha+\mu)u} + \\ + \frac{\alpha}{\beta+\mu}\,\frac{\mu}{\alpha-\mu-\beta}\left(1+\frac{\mu}{\alpha-\beta}\,\frac{\alpha}{\beta}\,\frac{\mu+3\beta-2\alpha}{\mu+\beta-2\alpha}\right) H(2,1,t-u)e^{-(\beta+\mu)u}\Big).$$

With account of these relations, $P(i,t)$ has the forms

$$P(0,t) = e^{-(3\alpha+4\beta+2\mu)t}\times \\ \times\Big(1+\int_0^t\int_0^s e^{(3\alpha+4\beta+2\mu)s}\Big(H(0,1,s-u)e^{-2\alpha u} + \frac{\mu}{\alpha-\mu} H(1,1,s-u)e^{-(\alpha+\mu)u} + \\ + \frac{\mu}{2\alpha-\mu+\beta}\,\frac{\mu}{\alpha-\beta} H(2,1,s-u)e^{-(\alpha+\mu)u}\Big)g(u)\,du\,ds,$$

$$P(1,t) = e^{-(3\alpha+4\beta+2\mu)t}\times \\ \times\int_0^t\int_0^s e^{(3\alpha+4\beta+2\mu)s}\Big(H(1,1,s-u)e^{-(\alpha+\mu)u} + \frac{\mu}{\alpha-\beta} H(2,1,s-u)e^{-(\beta+\mu)u}\Big)g(u)\,du\,ds,$$

$$P(2,t) = e^{-(3\alpha+4\beta+2\mu)t}\int_0^t\int_0^s e^{(3\alpha+4\beta+2\mu)s} H(2,1,s-u)e^{-(\beta+\mu)u}g(u)\,du\,ds. \tag{6}$$

In expressions of $P(i,t)$, $i=0,1,2$, and $q(i,j,t,u)$ the functions $H(i,t-u)$ are unknown. They can be easily expressed if we equate the functions $q(i,j,t,u)$ (6) at $u=0$ to the boundary conditions (3).

3. Conclusion

In this paper we present and study a mathematical model of redundant technical system with arbitrarily distributed repair time and exponentially distributed replacement time. A mathematical model for dependability and performance analysis is constructed and studied. An explicit solution for transient probabilities in terms of matrix exponential is obtained.

Using probabilistic considerations and matrix exponential techniques, we obtain an explicit solution for transient probabilities. The direction of future research is the consideration of the behavior of this system for different distribution functions of the renewal time.

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