Journal of Mathematical Sciences, Vol. 237, No. 5, March, 2019

NUMERICAL ANALYSIS OF RETRIAL QUEUEING SYSTEMS WITH CONFLICT OF CUSTOMERS AND AN UNRELIABLE SERVER

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In this paper a closed retrial queueing system is considered with a finite number of customers. If an arriving (primary or secondary) request finds the server busy, two modes are possible: the job is transferred to the orbit (no conflict) or the job under service is interrupted and both of them are transferred to the orbit (conflict). Jobs in the orbit can retry reaching the server after a random time. The unreliable case where the server is subject to breakdown is also investigated. These types of systems can be solved by numerical, asymptotical, and simulation methods. The novelty of the investigations is that it provides a new approach to an algorithmic solution for calculating the steady-state probabilities of the system. With the help of these probabilities the main performance measures can be computed. Several sample examples illustrate the effect of different parameters on the distribution on requests in the system.

1. Introduction

In real life situations a large variety of problems can be modeled and simulated with the help of retrial queueing systems (RQ-systems). A RQ-system can be described by the following characteristics: when an arriving job from the outside world (from the sources) or from the queue of the system finds the server busy, joins the orbit, and, after a random, usually exponentially distributed time, retries to reach the server again. The orbit is assumed to be infinitely large and jobs keep retrying until they are served. The main characteristic application fields of RQ-systems are the telecommunication systems, call centers, computer networks, telephone switching systems, and recently smart city networks, etc. Infinite source models have been investigated and applied by many authors, but in several cases the models with finite number of sources are more appropriate to describe the system behavior. The most typical examples are the mobile networks, sensor networks, smart city networks, and cognitive radio systems. The random and multiple access protocols for these type of systems were investigated, for example in [3, 9].

Unfortunately, the reliable operation of the considered systems cannot be assumed. They are subjected to breakdowns, which is why this situation has to be investigated. Modeling the system involves random server breakdowns and repairs. A nonreliable operation of the systems has a great influence on the system characteristics and performance measures. Finite-source RQ-systems with server breakdowns and repairs were investigated in several recent papers, for example in [2, 6, 7, 14, 15].

In the present paper we consider an $M/M/1//N$ retrial queueing system with conflict (collisions) of customers. In an unsynchronized communication system with a limited number of resources, e.g., communication channels, there is a significant chance of collision of requests. In this case the transmission is lost and the interrupted customers need to be retransmitted; consequently the performance of the system is sub-optimal. There is great importance in developing methods, procedures, and protocols which are able to prevent the system from customer conflicts or at least try to optimize the performance. In this direction some recent results can be found in [1, 4, 8, 10–13].

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Proceedings of the XXXV International Seminar on Stability Problems for Stochastic Models, Debrecen, Hungary, August 25–29, 2017. Part I

Our aim is to propose a new algorithmic solution approach to the calculation of the steady-state probabilities of the system. Using these probabilities, the main performance measures can be computed. Several examples illustrate the effect of different parameters on the distribution on requests in the system.

2. System model

Let us consider a finite source closed retrial $M/M/1//N$-type queuing system. This Kendall's notation means that the system has one server and the number of sources is N. In this paper two working characteristics of the server are distinguished:

- The server is reliable, that is, there are no breakdowns during its operation.
- The server is nonreliable, that is, it is subjected to random breakdowns after an exponentially distributed time. In the case of a busy server, the breakdown parameter is γ_0, while in the case of an idle server, the parameter is γ_1. It is assumed that the request under service is sent to the orbit. After the breakdown, the repair starts immediately. The distribution of the repair time is also exponential with parameter γ_2. During the down period of the server, the sources are supposed to be able to generate requests. These customers reach an unavailable server, and they will be transferred into the orbit. From the orbit, these requests retry reaching the server again after a exponentially distributed time with parameter σ/N. The customers keep reaching the server until they are served.

The sources generate a job (customer) towards the server. The inter-request times are exponentially distributed with parameter λ/N. After generating a customer, the source waits for a successful service. Until the end of the service of a job, the source cannot generate a new request. The generated calls reach the server. The server can be busy or idle. In an idle state, the customer enters into service immediately, and the service times are assumed to be exponentially distributed with parameter μ. In the case of a busy state of the server, two different working modes can be considered:

- No conflict: when an incoming job finds the server busy, it is transferred into the orbit. From the orbit the customer retries reaching the server again after an exponentially distributed time with parameter σ/N. The job keeps reaching the server until it is served.
- Conflict: when an arriving job finds the server busy, it is involved into collision with the customer under service and both customers are moved into the orbit. From the orbit the customers retry reaching the server again after exponentially distributed time with parameter σ/N. The jobs keep reaching the server until they are served. See the model in Fig. 1.

Let $i(t)$ be the number of sources in the service facility that are either in the orbit or under service, and let $k(t)$ denote the status of the server:

$$k(t) = \begin{cases} 0, \text{if the server is idle,} \\ 1, \text{if the server is busy,} \\ 2, \text{if the server is under repair.} \end{cases}$$

.

Denote by $\mathsf{P}(k(t) = k, i(t) = i) = \mathsf{P}_k(i,t)$ the probability that at a time t there are i sources in the "waiting" state and the server is in the state k. Under the above assumption the process $X(t) = \{k(t), i(t)\}$ is a 2-dimensional Markov chain with the state space $\{0,1,2\}x\{0,1,\dots,N\}$.

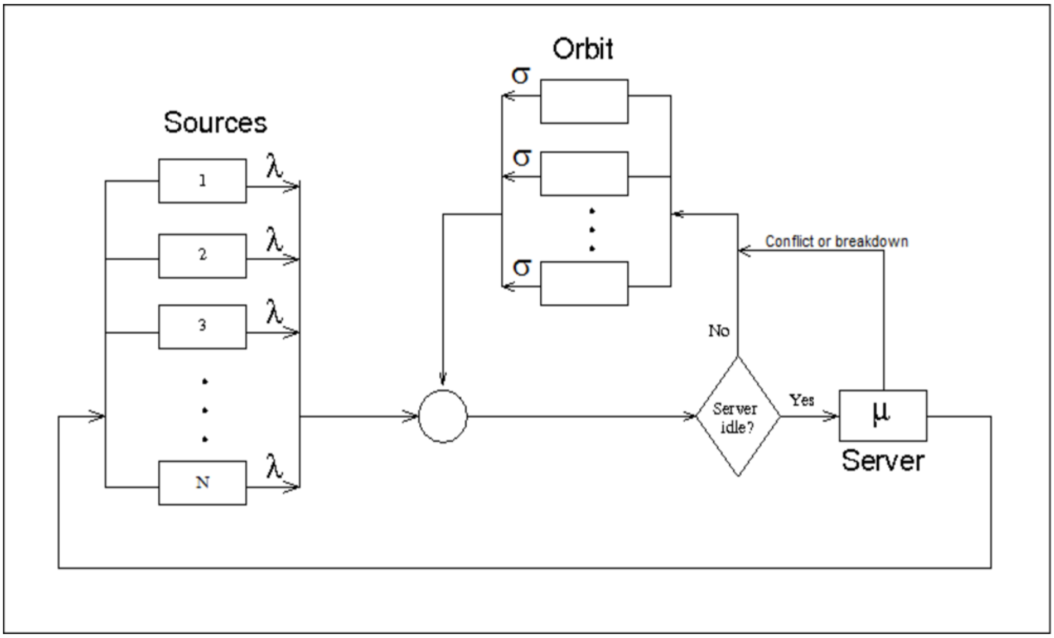

Fig. 1. System model.

When the submission (the service of a request) is successful, the request goes back to the source. All the random variables involved in the model construction are assumed to be totally independent from each other.

As it was described in [10,12], the Kolmogorov differential equations for the probabilities $\mathsf{P}_k(i,t)$ are

$$\frac{\partial \mathsf{P}_0(0,t)}{\partial t} = -(\lambda+\gamma_0)\mathsf{P}_0(0,t) + \mu\mathsf{P}_1(1,t) + \gamma_2\mathsf{P}_2(0,t),$$

$$\frac{\partial \mathsf{P}_1(1,t)}{\partial t} = -\left(\lambda\frac{N-1}{N} + \mu + \gamma_1\right)\mathsf{P}_1(1,t) + \lambda\mathsf{P}_0(0,t) + \frac{\sigma}{N}\mathsf{P}_0(1,t),$$

$$\frac{\partial \mathsf{P}_2(0,t)}{\partial t} = -(\lambda+\gamma_2)\mathsf{P}_2(0,t) + \gamma_0\mathsf{P}_0(0,t), \tag{1}$$

$$\begin{aligned}\frac{\partial \mathsf{P}_0(i,t)}{\partial t} &= -\left(\lambda\frac{N-1}{N} + \sigma\frac{i}{N} + \gamma_0\right)\mathsf{P}_0(i,t) + \mu\mathsf{P}_1(i+1,t) + \\ &+ \lambda\frac{N-i+1}{N}\mathsf{P}_1(i-1,t) + \sigma\frac{i-1}{N}\mathsf{P}_1(i,t) + \gamma_2\mathsf{P}_2(i,t),\end{aligned}$$

$$\frac{\partial \mathsf{P}_1(i,t)}{\partial t} = -\left(\lambda\frac{N-1}{N} + \sigma\frac{i-1}{N} + \gamma_1 + \mu\right)\mathsf{P}_1(i,t) + \lambda\frac{N-i+1}{N}\mathsf{P}_0(i-1,t) + \sigma\frac{i}{N}\mathsf{P}_0(i,t),$$

$$\frac{\partial \mathsf{P}_2(i,t)}{\partial t} = -\left(\lambda\frac{N-1}{N} + \gamma_2\right)\mathsf{P}_2(i,t) + \gamma_0\mathsf{P}_0(i,t) + \gamma_1\mathsf{P}_1(i,t) + \lambda\frac{N-i+1}{N}\mathsf{P}_2(i-1,t).$$

Since $X(t) = \{k(t), i(t)\}$ is a finite Markov chain, it can be assumed that it operates in the steady-state mode, that is, $\mathsf{P}_k(i,t) = \mathsf{P}_k(i)$. Hence, the steady-state Kolmogorov-equations can be written as

$$-(\lambda+\gamma_0)\mathsf{P}_0(0) + \mu\mathsf{P}_1(1) + \gamma_2\mathsf{P}_2(0) = 0,$$

$$-\left(\lambda\frac{N-1}{N}+\mu+\gamma_1\right)\mathsf{P}_1(1)+\lambda\mathsf{P}_0(0)+\frac{\sigma}{N}\mathsf{P}_0(1)=0,$$

$$-(\lambda+\gamma_2)\mathsf{P}_2(0)+\gamma_0\mathsf{P}_0(0)=0, \tag{2}$$

$$-\left(\lambda\frac{N-1}{N}+\sigma\frac{i}{N}+\gamma_0\right)\mathsf{P}_0(i)+\mu\mathsf{P}_1(i+1)+\lambda\frac{N-i+1}{N}\mathsf{P}_1(i-1)+\sigma\frac{i-1}{N}\mathsf{P}_1(i)+\gamma_2\mathsf{P}_2(i)=0,$$

$$-\left(\lambda\frac{N-1}{N}+\sigma\frac{i-1}{N}+\gamma_1+\mu\right)\mathsf{P}_1(i)+\lambda\frac{N-i+1}{N}\mathsf{P}_0(i-1)+\sigma\frac{i}{N}\mathsf{P}_0(i)=0,$$

$$-\left(\lambda\frac{N-1}{N}+\gamma_2\right)\mathsf{P}_2(i)+\gamma_0\mathsf{P}_0(i)+\gamma_1\mathsf{P}_1(i)+\lambda\frac{N-i+1}{N}\mathsf{P}_2(i-1)=0.$$

Note that if all of the γ_2 parameters and P_2 probabilities are set to zero, we get the formulas for the system with conflict and reliable server.

3. Numerical solution

Based on the equations described in [10, 12], the following steps can be used for the recursive calculation of the steady-state probabilities $\mathsf{P}_k(i)$:

Step 1. Set the numerical values for parameters $N, \lambda, \mu, \sigma, \gamma_0, \gamma_1, \gamma_2$. The values of these parameters for different figures can be seen in Table 1.

Step 2. Use the reasonable assumption: $\mathsf{P}_1(0)=0$.

Step 3. From the third equation of (2), $\frac{\mathsf{P}_2(0)}{\mathsf{P}_0(0)}$ can be calculated as

$$\frac{\mathsf{P}_2(0)}{\mathsf{P}_0(0)}=\frac{\gamma_0}{\lambda+\gamma_2}.$$

Step 4. From the third equation of (2), $\frac{\mathsf{P}_1(1)}{\mathsf{P}_0(0)}$ can be obtained as

$$\frac{\mathsf{P}_1(1)}{\mathsf{P}_0(0)}=\frac{1}{\mu}\left((\lambda+\gamma_0)-\gamma_2\frac{\mathsf{P}_2(0)}{\mathsf{P}_0(0)}\right).$$

Step 5. For $1\leqslant i\leqslant N-1$ the following equations can be derived:

$$\frac{\mathsf{P}_0(i)}{\mathsf{P}_0(0)}=\frac{N}{i\sigma}\left\{\left(\lambda\left(\frac{N-i}{N}\right)+\sigma\frac{i-1}{N}+\mu+\gamma_1\right)\frac{\mathsf{P}_1(i)}{\mathsf{P}_0(0)}-\lambda\left(\frac{N-i+1}{N}\right)\frac{\mathsf{P}_0(i-1)}{\mathsf{P}_0(0)}\right\},$$

$$\frac{\mathsf{P}_2(i)}{\mathsf{P}_0(0)}=\frac{1}{\gamma_2+\lambda\left(\frac{N-1}{N}\right)}\left(\gamma_0\frac{\mathsf{P}_0(i)}{\mathsf{P}_0(0)}+\gamma_1\frac{\mathsf{P}_1(i)}{\mathsf{P}_0(0)}+\lambda\left(\frac{N-i+1}{N}\right)\frac{\mathsf{P}_2(i-1)}{\mathsf{P}_0(0)}\right),$$

$$\frac{\mathsf{P}_1(i+1)}{\mathsf{P}_0(0)}=$$

$$=\frac{1}{\mu}\left\{\left(\lambda\left(\frac{N-i}{N}\right)+\sigma\frac{i}{N}+\gamma_0\right)\frac{\mathsf{P}_0(i)}{\mathsf{P}_0(0)}-\lambda\left(\frac{N-i+1}{N}\right)\frac{\mathsf{P}_1(i-1)}{\mathsf{P}_0(0)}-\sigma\frac{i-1}{N}\frac{\mathsf{P}_1(i)}{\mathsf{P}_0(0)}-\gamma_2\frac{\mathsf{P}_2(i)}{\mathsf{P}_0(0)}\right\}.$$

Step 6. For $i=N$ the following formulas are valid:

$$\frac{\mathsf{P}_0(i)}{\mathsf{P}_0(0)}=\frac{1}{\sigma}\left\{\left(\sigma\frac{N-1}{N}+\mu+\gamma_1\right)\frac{\mathsf{P}_1(N)}{\mathsf{P}_0(0)}-\frac{\lambda}{N}\frac{\mathsf{P}_0(N-1)}{\mathsf{P}_0(0)}\right\},$$

$$\frac{\mathsf{P}_2(N)}{\mathsf{P}_0(0)} = \frac{1}{\gamma_2}\left(\gamma_0\frac{\mathsf{P}_0(N)}{\mathsf{P}_0(0)} + \gamma_1\frac{\mathsf{P}_1(N)}{\mathsf{P}_0(0)} + \frac{\lambda}{N}\frac{\mathsf{P}_2(N-1)}{\mathsf{P}_0(0)}\right).$$

Step 7. Using the normalization condition, $\mathsf{P}_0(0)$ can be determined:

$$\mathsf{P}_0(0) = \frac{1}{\sum\limits_{i=0}^{N}\left(\frac{\mathsf{P}_0(i)}{\mathsf{P}_0(0)} + \frac{\mathsf{P}_1(i)}{\mathsf{P}_0(0)} + \frac{\mathsf{P}_2(i)}{\mathsf{P}_0(0)}\right)}.$$

Step 8. Given the values of $\frac{\mathsf{P}_k(i)}{\mathsf{P}_0(0)}$ and $\mathsf{P}_0(0)$, the probabilities $\mathsf{P}_k(i),\ k = 0, 1, 2$, can be obtained.

Step 9. The one-dimensional marginal distribution can be obtained as

$$\mathsf{P}(i) = \mathsf{P}_0(i) + \mathsf{P}_1(i) + \mathsf{P}_2(i), \quad i = 0, 1, \dots, N.$$

The calculation was performed by a spreadsheet program, MS Excel. This software is comfortable for solving this type of recursive calculations. The parameters are given into specified cells so the effect of changing different parameters can be observed easily. For programming the formulas the absolute and relative cell references are useful methods. The running parameter, the number of sources (i), is set into a column and N can be arbitrarily large. When the calculations are performed by the MOSEL-2 tool (MOdeling Specification and Evaluation Language, see in [5]), we run into a strict limitation, namely the state space grows extremely fast, and, consequently the number of sources cannot exceed 200. In Excel we can go far above 200. Other advantages for using this spreadsheet are that the effect of parameter modifications can be seen immediately and the sets of the steady-state probabilities, both the two- and one-dimensional, are present in separate columns and can be used directly for further investigations.

Table 1. Numerical values of model parameters

Figure	**Model**	λ	μ	σ	N	γ_0	γ_1	γ_2
2	All*	1	1	5	100	0.1	0.1	1
3	Unrel.	0.5, 1, 2	1	5	100	0.1	0.1	1
4	Unrel.	1	1, 2, 5	5	100	0.1	0.1	1
5	Unrel.	1	1	0.1, 1, 5	100	0.1	0.1	1
6	Unrel.	1	1	5	100	0.1, 0.2, 2	0.1	1
7	Unrel.	1	1	5	100	0.1	0.1	0.1, 1, 5
8	Unrel.	1	1	5	100	0.1, 1	1, 0.1	1
9	Unrel.	1	1	5	250	0.1	0.1	1
10	Unrel.	1	1	5	100	0.01, 0.1, 1	0.1	1

In the “All” model, all three considered models are included into the corresponding graph: reliable with no conflict, reliable with conflict, and unreliable with conflict. The other figures were generated for unreliable server with conflict of customers.

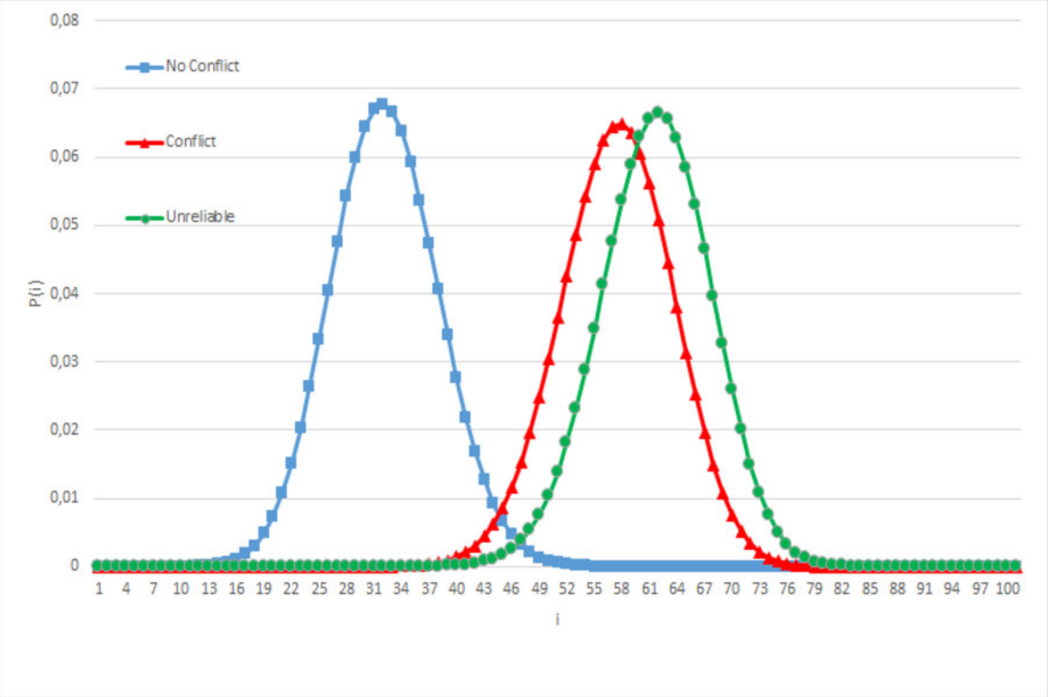

Fig. 2. Reliable no conflict, reliable with conflict, and unreliable with conflict.

In Fig. 2 the steady-state probabilities of three models are displayed: the basic reliable system with no conflict, reliable system with conflict, and unreliable system with conflict. In the no conflict case the expectation of states are lower than for the other cases. The probabilities of the states have the greatest mean in the unreliable system, as was expected.

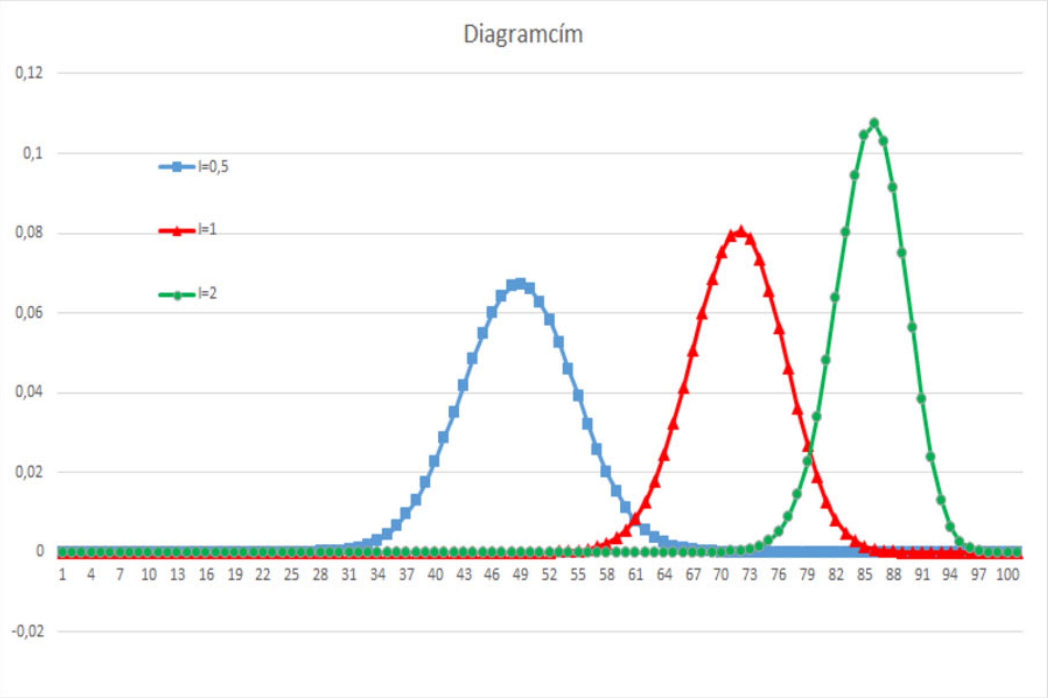

Fig. 3. Effect of generation rate λ.

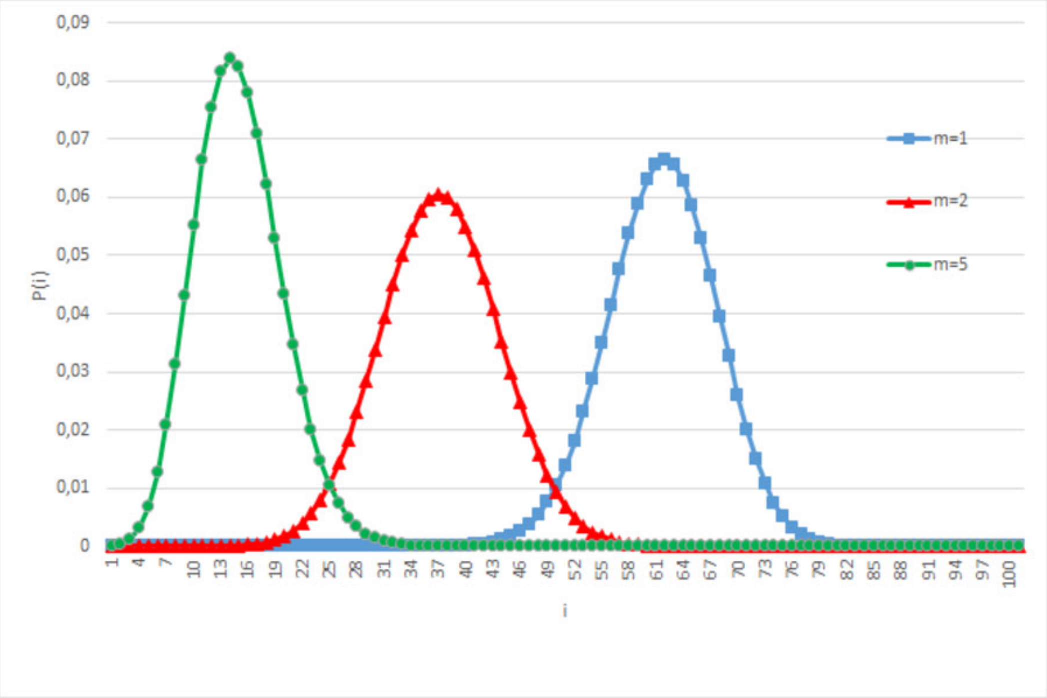

Fig. 4. Effect of service rate μ.

The effects of the customer generation rate, service rate, and retrial rate can be seen in Figs. 3–5, respectively.

The figures reflect the expected behavior: higher generation rates involve higher number of states; thus, the higher the mean, the lower the mean number of requests for higher service end retrial rates, as was expected.

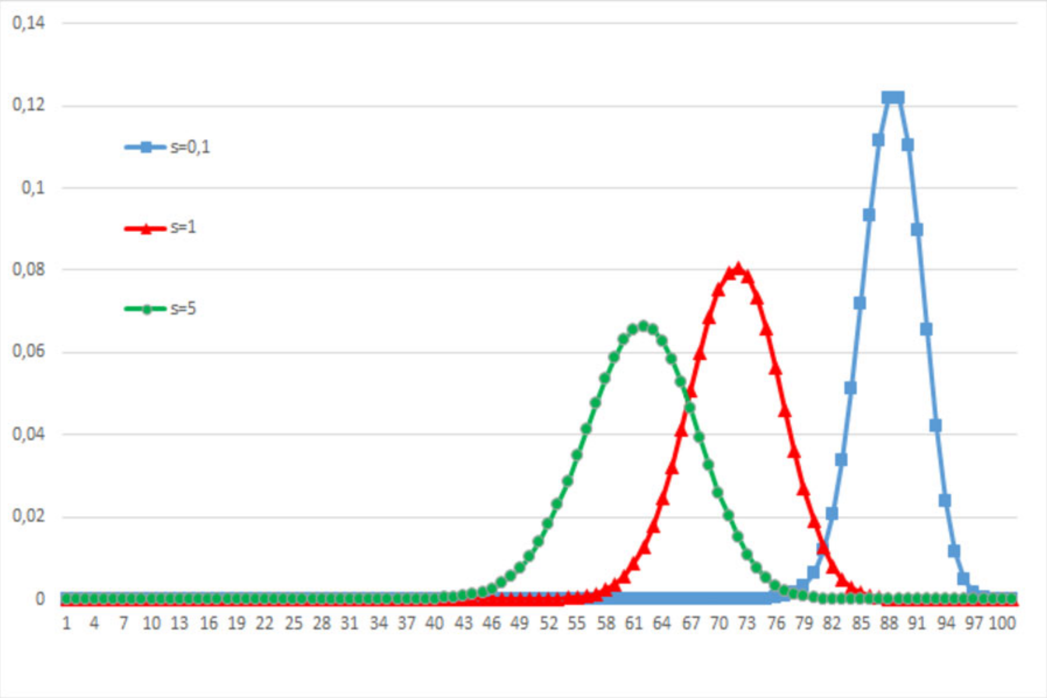

Fig. 5. Effect of retrial rate σ.

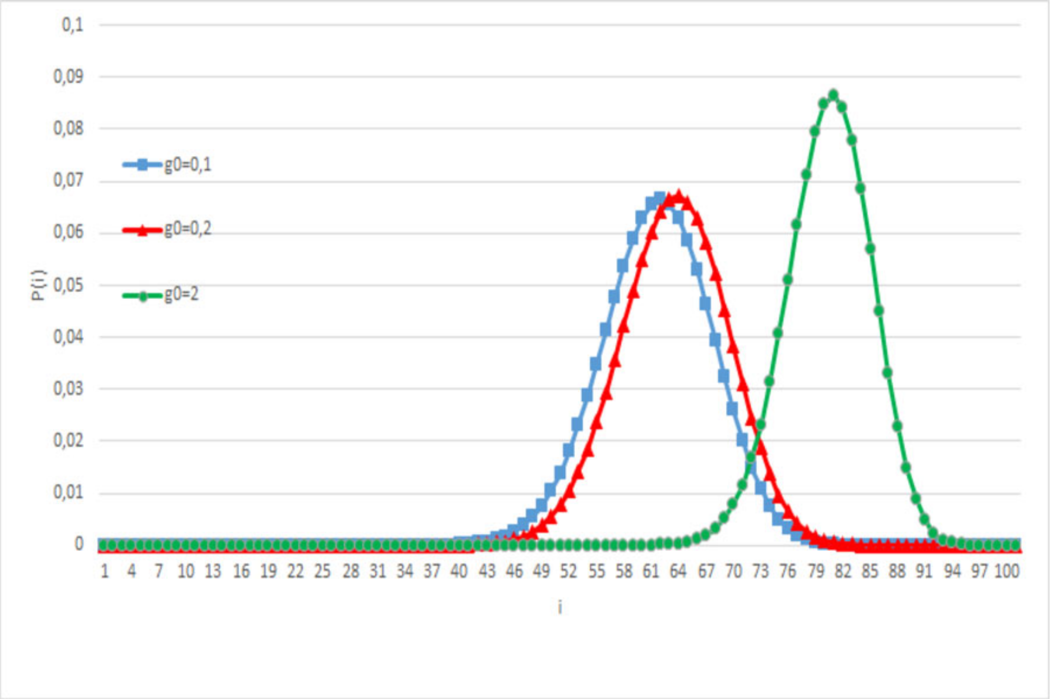

Fig. 6. Effect of failure rate γ_0.

In Figs. 6 and 7 the effect of failure and repair rates is displayed. It can be observed that for significantly higher idle failure rate the mean value of customers in the system is much larger than for the other two cases. The change of the repair rate provides a similar result. The slower the repair is, the higher the number of customer. The parameter scenario for γ_1 is not presented here because the result is very similar for scenario γ_0.

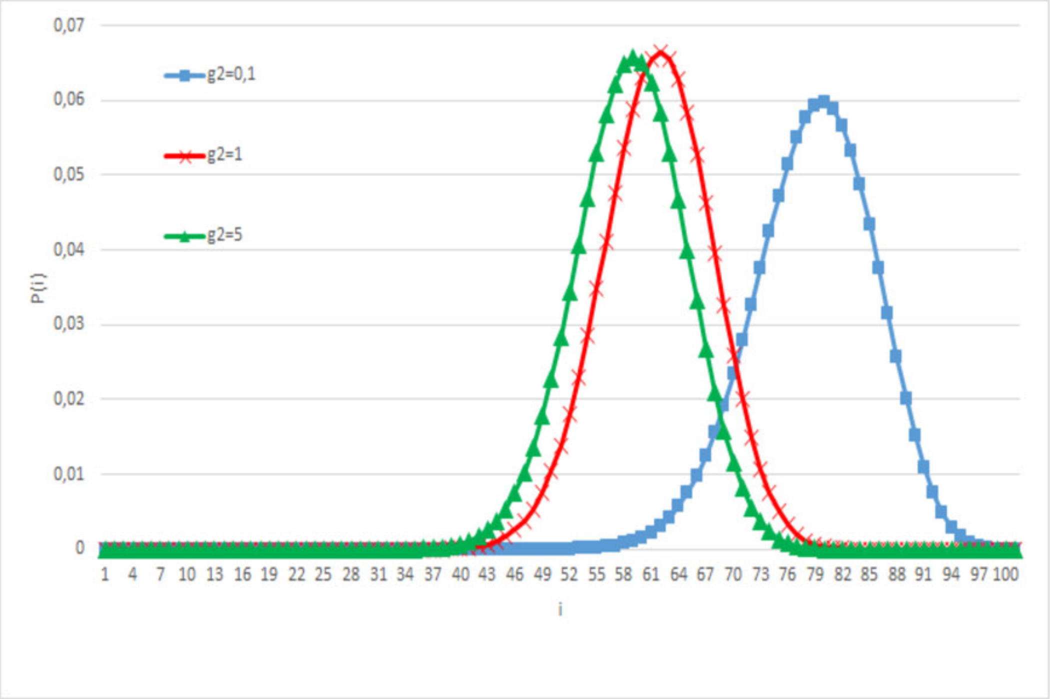

Fig. 7. Effect of repair rate γ_2.

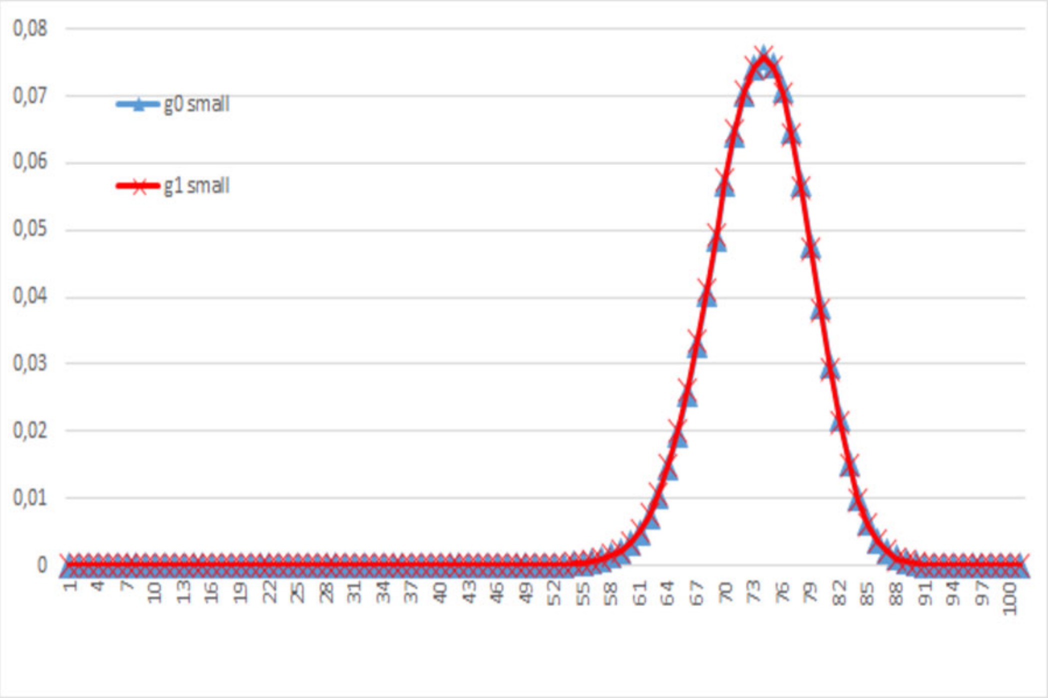

Fig. 8. Effect of failure rates γ_0 vs. γ_1.

Figure 8 shows the small γ_0 large γ_1 vs. large γ_0 small γ_1 comparison. Because of the similarity mentioned above, there is no difference between the two cases.

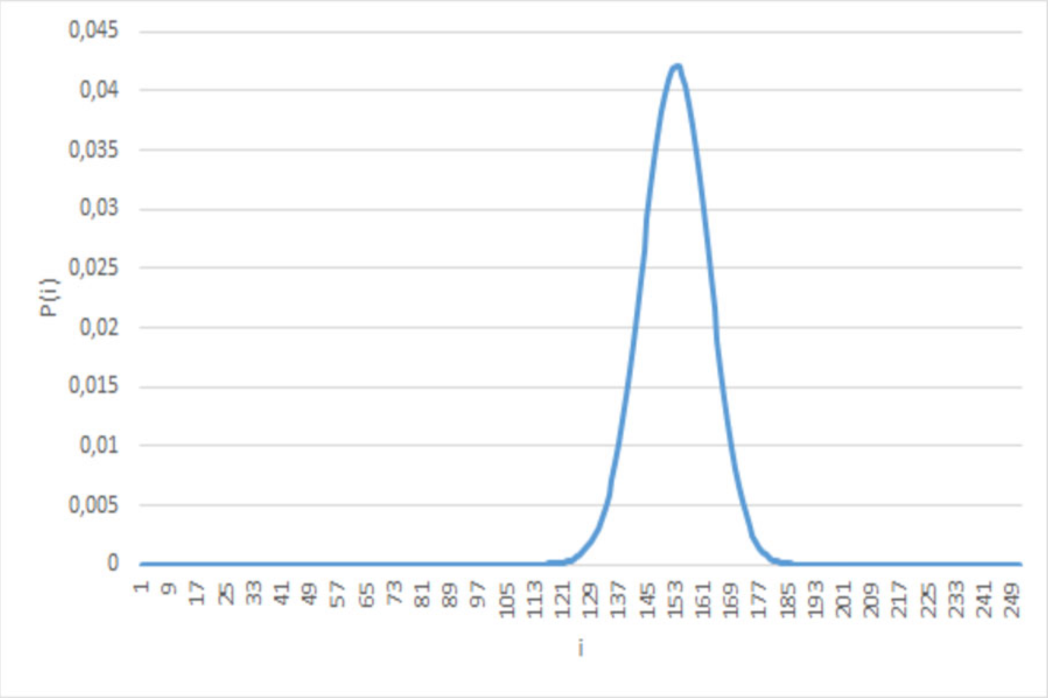

Fig. 9. Number of sources: 250.

Figure 9 demonstrates the advantage of the numerical calculation compared with the MOSEL-calculation. MOSEL has a strict state space limitation while this recursive numerical method can be performed far beyond that point.

In Fig. 10 the result of the comparison between the numerical and MOSEL-calculation is displayed. The empirical distribution functions are calculated by cumulating the steady-state probabilities and the Kolmogorov distance between distribution functions is applied.

The Kolmogorov-distance is defined as

$$\Delta = \max_{0\leqslant k\leqslant N}\left|\sum_{i=0}^{k}\mathsf{P}_{Num}(i)-\sum_{i=0}^{k}\mathsf{P}_{Mos}(i)\right|.$$

The numerical results for different values of idle failure rate γ_0 are shown in Table 2.

Table 2. The Kolmogorov distances

$\boldsymbol{\gamma_0}$	0.01	0.1	1
$\boldsymbol{\Delta}$	1.1053E−06	1.11192E−06	9.16013E−07

It can be stated that the results of two different calculations (numerical and MOSEL) are almost identical.

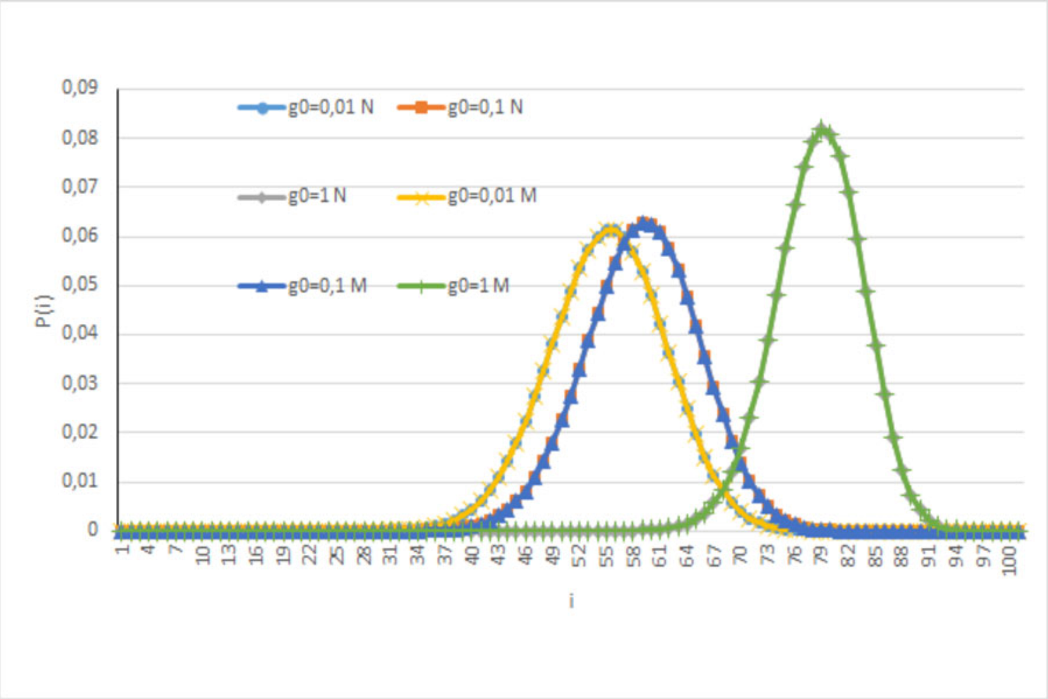

Fig. 10. Numerical calculations vs. MOSEL-2.

4. Conclusions

In this paper a finite-source retrial queueing model was introduced. Mainly the cases of an unreliable server and the conflict of customers were investigated, but some scenarios considered the reliable and no conflict cases as well. The goal of the paper was to provide an alternative solution for the tool-supported (MOSEL) numeric calculations of the steady-state probabilities. A robust software package, the MS Excel, proved to be useful and efficient for the solution. The main advantage of an algorithmic approach is that there is no memory limitations, and the values of the system probabilities are immediately ready for further use and investigations. The results of calculations were compared with the results of the

MOSEL-output. With the help of the Kolmogorov distance the two sets of probabilities were found to be almost identical.

Acknowledgments

The work/publication of J. Sztrik is supported by the EFOP-3.6.1-16-2016-00022 project. The project is co-financed by the European Union and the European Social Fund.

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