

ASYMPTOTIC ANALYSIS OF COMPLEX, MARKOV-MODULATED COMPUTER AND COMMUNICATION SYSTEMS

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Abstract

This paper deals with a First-Come, First-Served (FCFS) queueing model to analyse the asymptotic behaviour of a heterogeneous finite-source computer and communication systems with a single processor. Each source and the processor are supposed to operate in independent random environment, respectively, allowing the arrival and service processes to be Markov-modulated ones. Each message is characterized by its own exponentially distributed source and processing time with parameter depending on the state of the corresponding environment, that is, the arrival and service rates are subject to random fluctuations. Assuming that the arrival rates of the messages are many times greater than their service rates ("fast" arrival), it is shown that the time to the first system failure converges in distribution, under appropriate norming, to an exponentially distributed random variable. Some simple examples are considered to illustrate the effectiveness of the method proposed by comparing the approximate results to the exact ones.

Keywords: finite-source, queueing, communication system, Markov-modulation, random environment, "fast" arrival, reliability, weak convergence

1 Introduction

Performance evaluation of information system development has become more complex as the size and complexity of the system has increased, (c.f., Harrison [7], Mitrani [11], Takagi [17]). Reliability is certainly one of the most important characteristics for communication networks. The measure of greatest interest is the distribution of the time to the first system failure. It is well-known that the majority of the problems can be treated by the help of Semi-Markov Processes (SMP). Since the failure-free operation time of the system corresponds to sojourn time problems we can use the results obtained for SMP. If the exit from a given subset of the state space is a "rare" event, that is, it occurs with a small probability it is natural to investigate the asymptotic behaviour of the sojourn time in that subspace, see Kovalenko [10], Gertsbakh [5-6], Keilson [8]. Realistic consideration of certain stochastic systems, however, often requires the introduction of a random environment, sometimes referred as to Markov-modulation, where system parameters are subjected to randomly occurring fluctuations. This situation may be attributed to certain changes in the physical environment such as weather, or sudden personal changes and work load alterations. Gaver *et al.* [4] proposed an efficient computational approach for the analysis of a generalised structure involving finite state space birth-and-death processes in a Markovian environment. Stern and Elwalid [12] has used Markov-modulated processes

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for analysing some infinite source information systems. This paper deals with a First-Come, First-Served (FCFS) queueing model to analyse the asymptotic behaviour of a heterogeneous finite-source communication system with a single processor. The sources and the processor are supposed to operate in independent random environments, respectively. Each message is characterized by its own exponentially distributed source and processing time with parameter depending on the state of the corresponding environment. Assuming that the arrival rates of the messages are many times greater than their service rates ("fast" arrival), it is shown that the time to the first system failure converges in distribution, under appropriate norming, to an exponentially distributed random variable. Some simple examples are considered to illustrate the effectiveness of the method proposed comparing the approximate characteristics to the exact ones. This paper generalizes the results of Sztrik and Lukashuk [15], Sztrik and Rigo [16], where the sources are homogeneous and the whole system is governed by a single random environment, two random environments, respectively. The technique used here is similar to the one applied in Kouvatsos [9], and Sztrik [13,14].

2 Preliminary Results

In this section a brief survey is given of the most related theoretical results, mainly due to Anisimov, see Anisimov [1-3], to be applied later on.

Let $(X_\varepsilon(k), k \geq 0)$ be a Markov chain with state space

$$\bigcup_{q=0}^{m+1} X_q, \quad X_i \cap X_j = \emptyset, \quad i \neq j,$$

defined by the transition matrix $(p_\varepsilon(i^{(q)}, j^{(z)}))$ satisfying the following conditions:

1. $p_\varepsilon(i^{(0)}, j^{(0)}) \rightarrow p_0(i^{(0)}, j^{(0)})$, as $\varepsilon \rightarrow 0$, $i^{(0)}, j^{(0)} \in X_0$, and matrix $P_0 = (p_0(i^{(0)}, j^{(0)}))$ is irreducible;
2. $p_\varepsilon(i^{(q)}, j^{(q+1)}) = \varepsilon \alpha^{(q)}(i^{(q)}, j^{(q+1)}) + o(\varepsilon)$, $i^{(q)} \in X_q, j^{(q+1)} \in X_{q+1}$;
3. $p_\varepsilon(i^{(q)}, f^{(q)}) \rightarrow 0$, as $\varepsilon \rightarrow 0$, $i^{(q)}, f^{(q)} \in X_q, q \geq 1$;
4. $p_\varepsilon(i^{(q)}, f^{(z)}) \equiv 0$, $i^{(q)} \in X_q, f^{(z)} \in X_z, z - q \geq 2$.

In the sequel the set of states X_q is called the q -th level of the chain, $q = 1, \dots, m+1$. Let us single out the subset of states

$$\langle \alpha_m \rangle = \bigcup_{q=0}^m X_q.$$

Denote by $\{\pi_\varepsilon(i^{(q)}), i^{(q)} \in X_q\}$, $q = 1, \dots, m$ the stationary distribution of a chain with transition matrix

$$\left(\frac{p_\varepsilon(i^{(q)}, j^{(z)})}{1 - \sum_{k^{(m+1)} \in X_{m+1}} p_\varepsilon(i^{(q)}, k^{(m+1)})} \right), \quad i^{(q)} \in X_q, j^{(z)} \in X_z, q, z \leq m.$$

Furthermore denote by $g_\varepsilon(\langle \alpha_m \rangle)$ the steady state probability of exit from $\langle \alpha_m \rangle$, that is

$$g_\varepsilon(\langle \alpha_m \rangle) = \sum_{i^{(m)} \in X_m} \pi_\varepsilon(i^{(m)}) \sum_{j^{(m+1)} \in X_{m+1}} p_\varepsilon(i^{(m)}, j^{(m+1)}).$$

Denote by $\{\pi_0(i^{(0)}), i^{(0)} \in X_0\}$ the stationary distribution corresponding to P_0 and let

$$\bar{\pi}_0 = \left\{ \pi_0(i^{(0)}), i^{(0)} \in X_0 \right\}, \quad \bar{\pi}_\varepsilon^{(q)} = \left\{ \pi_\varepsilon(i^{(q)}), i^{(q)} \in X_q \right\},$$

be row vectors. Finally, let the matrix

$$A^{(q)} = \left(\alpha^{(q)}(i^{(q)}, j^{(q+1)}) \right), \quad i^{(q)} \in X_q, \quad j^{(q+1)} \in X_{q+1}, \quad q = 0, \dots, m,$$

defined by condition 2.

Conditions 1–4 enable us to compute the main terms of the asymptotic expression for $\bar{\pi}_\varepsilon^{(q)}$ and $g_\varepsilon(\langle \alpha_m \rangle)$. Namely, we obtain

$$\begin{aligned} \bar{\pi}_\varepsilon^{(q)} &= \varepsilon^q \bar{\pi}_0 A^{(0)} A^{(1)} \dots A^{(q-1)} + o(\varepsilon^q), \quad q = 1, \dots, m, \\ g_\varepsilon(\langle \alpha_m \rangle) &= \varepsilon^{m+1} \bar{\pi}_0 A^{(0)} A^{(1)} \dots A^{(m)} \underline{1} + o(\varepsilon^{m+1}), \end{aligned} \tag{1}$$

where $\underline{1} = (1, \dots, 1)^*$ is a column vector. Let $(\eta_\varepsilon(t), t \geq 0)$ be a SMP given by the embedded Markov chain $(X_\varepsilon(k), k \geq 0)$ satisfying conditions 1–4. Let the times $\tau_\varepsilon(j^{(s)}, k^{(z)})$ – transition times from state $j^{(s)}$ to state $k^{(z)}$ – fulfil the condition

$$E \exp \left\{ i\theta \beta_\varepsilon \tau_\varepsilon(j^{(s)}, k^{(z)}) \right\} = 1 + a_{jk}(s, z, \theta) \varepsilon^{m+1} + o(\varepsilon^{m+1}), \quad (i^2 = -1),$$

where β_ε is some normalizing factor. Denote by $\Omega_\varepsilon(m)$ the instant at which the SMP reaches the $(m+1)$ -th level for the first time, exit time from $\langle \alpha_m \rangle$, provided $\eta_\varepsilon(0) \in \langle \alpha_m \rangle$. Then we have:

Theorem 1 (cf. Anisimov [2]) *If the above conditions are satisfied then*

$$\lim_{\varepsilon \rightarrow 0} E \exp \{ i\theta \beta_\varepsilon \Omega_\varepsilon(m) \} = (1 - A(\theta))^{-1}$$

where

$$A(\theta) = \frac{\sum_{j^{(0)}, k^{(0)} \in X_0} \pi_0(j^{(0)}) p_0(j^{(0)}, k^{(0)}) a_{jk}(0, 0, \theta)}{\bar{\pi}_0 A^{(0)} A^{(1)} \dots A^{(m)} \underline{1}}$$

Corollary 1 *In particular, if $a_{jk}(s, z, \theta) = i\theta m_{jk}(s, z)$ then the limit is an exponentially distributed random variable with mean*

$$\frac{\sum_{j^{(0)}, k^{(0)} \in X_0} \pi_0(j^{(0)}) p_0(j^{(0)}, k^{(0)}) m_{jk}(0, 0)}{\bar{\pi}_0 A^{(0)} A^{(1)} \dots A^{(m)} \underline{1}}.$$

3 The Queueing Model

Consider a finite-source communication system with N heterogeneous sources and a single processor. The sources and the receiver operate in independent random environments. The environmental changes are reflected in the values of the acces and service rates that prevail at any point of time. The main objective is to adapt these parameters to respond to random changes effectively and thus maintain derived level of system performance.

Source p is assumed to operate in a random environment governed by an ergodic Markov chain $(\xi_p(t), t \geq 0)$ with state space $(1, \dots, r_p)$ and with transition density matrix

$$\left(a_{i_p j_p}^{(p)}, i_p, j_p = 1, \dots, r_p, a_{i_p i_p}^{(p)} = \sum_{k \neq i_p} a_{i_p k}^{(p)} \right).$$

Whenever the environmental process $\xi_p(t)$ is in state i_p the probability that the source generates a request in the time interval $(t, t+h)$ is $\lambda_p(i_p, \varepsilon) + o(h)$, $p = 1, \dots, N$. Each message is transmitted to a receiver where the service immediately starts if the processor is idle, otherwise a queueing line is formed. The service discipline is First-Come, First-Served (FCFS). The receiver is also supposed to operate in a random environment governed by an ergodic Markov chain $(\xi(t), t \geq 0)$ with state space $(1, \dots, r_{N+1})$ and with transition density matrix

$$\left(a_{i_{N+1}j_{N+1}}^{(N+1)}, i_{N+1}, j_{N+1} = 1, \dots, r_{N+1}, a_{i_{N+1}i_{N+1}}^{(N+1)} = \sum_{k \neq i_{N+1}} a_{i_{N+1}k}^{(N+1)} \right).$$

Whenever the environmental process $\xi_{N+1}(t)$ is in state i_p the probability that the service of message p is completed in time interval $(t, t+h)$ is $\mu_p(i_{N+1})h + o(h)$. If a given source has sent a message it stays idle and it can not generate another one. After being serviced each message immediately returns to its source which hence becomes active. All random variables involved here and the random environments are supposed to be independent of each other.

In practical applications it is very important to know the distribution of time until the receiver becomes empty, which is actually the busy period of the processor. For example, if it is a satellite, an aircraft or any flying object which needs orientation.

Let us consider the system under the assumption of "fast" arrivals, i.e., $\lambda_p(i_p, \varepsilon) \rightarrow \infty$ as $\varepsilon \rightarrow 0$. For simplicity let $\lambda_p(i_p, \varepsilon) = \lambda_p(i_p)/\varepsilon$, $p = 1, 1, \dots, N$.

Denote by $Y_\varepsilon(t)$ the number of active sources at time t , and let

$$\Omega_\varepsilon(m) = \inf\{t : t > 0, Y_\varepsilon(t) = m + 1 / Y_\varepsilon(0) \leq m\},$$

that is, the instant at which the number of active sources reaches the $(m+1)$ -th level for the first time, provided that at the beginning their number is not greater than m , $m = 0, \dots, N-1$. In the following $\Omega_\varepsilon(m)$ is referred to as the time to the first system failure. In particular, for $m = N-1$ we get the case when the processor becomes idle, i.e. $\Omega_\varepsilon(N-1)$ is the busy period length of the receiver, or processor.

Denote by $\pi_0(i_1, i_2, \dots, i_N, i_{N+1} : 0; k_1, \dots, k_N)$ the steady-state probability that random environment $\xi_p(t)$ is in state i_p , $p = 1, \dots, N+1$, there is no active source and the order of arrival of messages to the processor is (k_1, \dots, k_N) . Similarly, denote by $\pi_0(i_1, i_2, \dots, i_N, i_{N+1} : 1; k_2, \dots, k_N)$ the steady-state probability that the p -th random environment is in state i_p , $p = 1, \dots, N+1$, source k_1 is active and the other sources sent their messages in the order (k_2, \dots, k_N) . Clearly, $(k_s, \dots, k_N) \in V_N^{N-s+1}$, $s = 1, 2$ where V_N^{N-s+1} denotes the set of all variations of order $N-s+1$ of elements $1, \dots, N$. Now we have:

Theorem 2 *For the system in question under the above assumptions, independently of the initial state, the distribution of the normalized random variable $\varepsilon^m \Omega_\varepsilon(m)$ converges weakly to an exponentially distributed random variable with parameter*

$$\Lambda = \sum_{i_1=1}^{r_1} \dots \sum_{i_{N+1}=1}^{r_{N+1}} \sum_{(k_1, \dots, k_N) \in V_N^N} \pi_0(i_1, \dots, i_{N+1} : 1; k_2, \dots, k_N) \\ \times \frac{\mu_{k_2}(i_{N+1})}{\lambda_{k_1}(i_{k_1})} \times \frac{\mu_{k_3}(i_{N+1})}{\lambda_{k_1}(i_{k_1}) + \lambda_{k_2}(i_{k_2})} \times \dots \times \frac{\mu_{k_{m+1}}(i_{N+1})}{\lambda_{k_1}(i_{k_1}) + \dots + \lambda_{k_m}(i_{k_m})} \frac{1}{D},$$

where

Proof. Let us introduce the following stochastic process

$$Z_\varepsilon(t) = (\xi_1(t), \dots, \xi_{N+1}(t) : Y_\varepsilon(t); \beta_1(t), \dots, \beta_{Y_\varepsilon(t)}),$$

where $\beta_1(t), \dots, \beta_{Y_\varepsilon(t)}$ denote the indices of the messages in order of their arrival at the processor. It is easy to see that $(Z_\varepsilon(t), t \geq 0)$ is a multi-dimensional Markov chain with rather complex state space

$$E = ((i_1, \dots, i_{N+1} : s; k_1, \dots, k_{N-s}), i_p = 1, \dots, r_p, p = 1, \dots, N+1, \\ (k_1, \dots, k_{N-s}) \in V_N^{N-s}, s = 0, \dots, N),$$

where $k_0 = \{0\}$ by definition. Furthermore, let

$$\langle \alpha_m \rangle = ((i_1, \dots, i_{N+1} : s; k_1, \dots, k_{N-s}), i_p = 1, \dots, r_p, p = 1, \dots, N+1, \\ (k_1, \dots, k_{N-s}) \in V_N^{N-s}, s = 0, \dots, m)$$

be a subset of the states. Hence our aim is to determine the distribution of the first exit time of $Z_\varepsilon(t)$ from $\langle \alpha_m \rangle$, provided that $Z_\varepsilon(t) \in \langle \alpha_m \rangle$. It can easily be verified that the transition probabilities in any time interval $(t, t+h)$ are the following:

$$(i_1, \dots, i_{N+1} : s; k_1, \dots, k_{N-s}) @ > h \gg \\ \left\{ \begin{array}{ll} (i_1, \dots, j_p, \dots, i_{N+1} : s; k_1, \dots, k_{N-s}) & a_{i_p j_p}^{(p)}, s = 0, \dots, N, j_p \neq i_p, p = 1, \dots, N+1 \\ (i_1, \dots, i_{N+1} : s+1; k_2, \dots, k_{N-s}) & \mu_{k_1}(i_{N+1})h + o(h), s = 0, \dots, N-1 \\ (i_1, \dots, i_{N+1} : s-1; k_1, \dots, k_{N-s+1}) & \lambda_{k_{N-s}}(i_{k_{N-s}})/\varepsilon + o(h), s = 1, \dots, N. \end{array} \right.$$

In addition, the sojourn time $\tau_\varepsilon(i_1, \dots, i_{N+1} : s; k_1, \dots, k_{N-s})$ of $Z_\varepsilon(t)$ in state $(i_1, \dots, i_{N+1} : s; k_1, \dots, k_{N-s})$ is exponentially distributed with parameter

$$\gamma_\varepsilon(i_1, \dots, i_{N+1} : s; k_1, \dots, k_{N-s}) = \\ \left\{ \begin{array}{l} a_{i_1 i_1}^{(1)} + \dots + a_{i_{N+1} i_{N+1}}^{(N+1)} + \mu_{k_1}(i_{N+1}), s = 0, \\ a_{i_1 i_1}^{(1)} + \dots + a_{i_{N+1} i_{N+1}}^{(N+1)} + \sum_{j \neq k_1, \dots, k_{N-s}} \lambda_j(i_j)/\varepsilon + \mu_{k_1}(i_{N+1}), s = 1, \dots, N-1, \\ a_{i_1 i_1}^{(1)} + \dots + a_{i_{N+1} i_{N+1}}^{(N+1)} + \sum_{j=1}^N \lambda_j(i_j)/\varepsilon, s = N. \end{array} \right.$$

Thus, the transition probabilities for the embedded Markov chain are

$$p_\varepsilon[(i_1, \dots, i_{N+1} : s; k_1, \dots, k_{N-s}), (i_1, \dots, j_p, \dots, i_{N+1} : s; k_1, \dots, k_{N-s})] \\ = \frac{a_{i_p j_p}^{(p)}}{\gamma_\varepsilon(i_1, \dots, i_{N+1} : s; k_1, \dots, k_{N-s})}, \quad \text{for } s = 0, \dots, N, p = 1, \dots, N+1, \\ p_\varepsilon[(i_1, \dots, i_{N+1} : s; k_2, \dots, k_{N-s}), (i_1, \dots, i_{N+1} : s+1; k_1, \dots, k_{N-s+1})] \\ = \frac{\mu_{k_1}(i_{N+1})}{\gamma_\varepsilon(i_1, \dots, i_{N+1} : s; k_1, \dots, k_{N-s})}, \quad \text{for } s = 0, \dots, N-1, \\ p_\varepsilon[(i_1, \dots, i_{N+1} : s; k_1, \dots, k_{N-s}), (i_1, \dots, i_{N+1} : s-1; k_1, \dots, k_{N-s-1})] \\ = \frac{\lambda_{k_{N-s}}(i_{k_{N-s}})/\varepsilon}{\gamma_\varepsilon(i_1, \dots, i_{N+1} : s; k_1, \dots, k_{N-s})}, \quad \text{for } s = 1, \dots, N,$$

As $\varepsilon \rightarrow 0$ implies

$$\begin{aligned}
& p_\varepsilon[(i_1, \dots, i_{N+1} : 0; k_1, \dots, k_N), (i_1, \dots, j_p, \dots, i_{N+1} : 0; k_1, \dots, k_N)] \\
&= \frac{a_{i_p j_p}^{(p)}}{a_{i_1 i_1}^{(1)} + \dots + a_{i_{N+1} i_{N+1}}^{(N+1)} + \mu_{k_1}(i_{N+1})}, \quad p = 1, \dots, N+1, \\
& p_\varepsilon[(i_1, \dots, i_{N+1} : s; k_1, \dots, k_{N-s}), (i_1, \dots, j_p, \dots, i_{N+1} : s; k_1, \dots, k_{N-s})] = o(1), \\
& \quad \text{for } s = 1, \dots, N, p = 1, \dots, N+1, \\
& p_\varepsilon[(i_1, \dots, i_{N+1} : 0; k_1, \dots, k_N), (i_1, \dots, i_{N+1} : 1; k_2, \dots, k_N)] \\
&= \frac{\mu_{k_1}(i_{N+1})}{a_{i_1 i_1}^{(1)} + \dots + a_{i_{N+1} i_{N+1}}^{(N+1)} + \mu_{k_1}(i_{N+1})}, \\
& p_\varepsilon[(i_1, \dots, i_{N+1} : s; k_1, \dots, k_{N-s}), (i_1, \dots, i_{N+1} : s+1; k_2, \dots, k_{N-s})] \\
&= \frac{\mu_{k_1}(i_{N+1})\varepsilon}{\sum_{p \neq k_1, \dots, k_{N-s}} \lambda_p(i_p)} (1 + o(1)), \quad s = 1, \dots, N-1.
\end{aligned}$$

This agrees with the conditions 1-4, but here the zero level is the set

$$\begin{aligned}
& ((i_1, \dots, i_{N+1} : s; k_1, \dots, k_{N-s}), \quad i_p = 1, \dots, r_p, \quad p = 1, \dots, N+1, \\
& \quad (k_1, \dots, k_{N-s}) \in V_N^{N-s}, \quad s = 0, 1),
\end{aligned}$$

and the q -th level is the set

$$\begin{aligned}
& ((i_1, \dots, i_{N+1} : q+1; k_1, \dots, k_{N-q-1}), \quad i_p = 1, \dots, r_p, \quad p = 1, \dots, N+1, \\
& \quad (k_1, \dots, k_{N-q-1}) \in V_N^{N-q-1}).
\end{aligned}$$

Since the level 0 in the limit forms an essential class, the probabilities

$$\pi_0(i_1, \dots, i_{N+1} : 0; k_1, \dots, k_N), \quad \pi_0(i_1, \dots, i_{N+1} : 1; k_1, \dots, k_{N-1}),$$

$i_p = 1, \dots, r_p, p = 1, \dots, N+1, (k_1, \dots, k_{N-s}) \in V_N^{N-s}, s = 0, 1$ satisfy the following system of equations

$$\begin{aligned}
\pi_0(j_1, j_2, \dots, j_{N+1} : 0; k_1, \dots, k_N) &= \sum_{i_1 \neq j_1} \pi_0(i_1, j_2, \dots, j_{N+1} : 0; k_1, \dots, k_N) \\
&\times a_{i_1 j_1}^{(1)} \left(a_{i_1 i_1}^{(1)} + a_{j_2 j_2}^{(2)} + \dots + a_{j_{N+1} j_{N+1}}^{(N+1)} + \mu_{k_1}(j_{N+1}) \right)^{-1} \\
&+ \sum_{i_2 \neq j_2} \pi_0(j_1, j_2, \dots, j_{N+1} : 0; k_1, \dots, k_N) \\
&\times a_{i_2 j_2}^{(2)} \left(a_{j_1 j_1}^{(1)} + a_{i_2 i_2}^{(2)} + \dots + a_{j_{N+1} j_{N+1}}^{(N+1)} + \mu_{k_1}(j_{N+1}) \right)^{-1} + \dots \\
&+ \sum_{i_{N+1} \neq j_{N+1}} \pi_0(j_1, j_2, \dots, i_{N+1} : 0; k_1, \dots, k_N) \\
&\times a_{i_{N+1} j_{N+1}}^{(N+1)} \left(a_{j_1 j_1}^{(1)} + a_{j_2 j_2}^{(2)} + \dots + a_{i_{N+1} i_{N+1}}^{(N+1)} + \mu_{k_1}(i_{N+1}) \right)^{-1} \\
&+ \pi_0(j_1, \dots, j_{N+1} : 1; k_1, \dots, k_{N-1})
\end{aligned} \tag{2}$$

$$\begin{aligned} \pi_0(j_1, \dots, j_{N+1} : 1; k_1, \dots, k_{N-1}) &= \pi_0(j_1, \dots, j_{N+1} : 0; k_N, k_1, \dots, k_{N-1}) \\ &\times \mu_{k_N}(j_{N+1}) \left(a_{j_1 j_1}^{(1)} + \dots + a_{j_{N+1} j_{N+1}}^{(N+1)} + \mu_{k_N}(j_{N+1}) \right)^{-1}. \end{aligned} \quad (3)$$

To apply (1) we need the solution of (2),(3) with normalizing condition

$$\begin{aligned} \sum_{p=1}^{N+1} \sum_{i_p=1}^{r_p} \sum_{(k_1, \dots, k_N)} \{ \pi_0(i_1, \dots, i_{N+1} : 0; k_1, \dots, k_N) \\ + \pi_0(i_1, \dots, i_{N+1} : 1; k_1, \dots, k_{N-1}) \} = 1. \end{aligned} \quad (4)$$

Suppose that we have this solution. Then by substituting it into (1) we get

$$\begin{aligned} g(\langle \alpha_m \rangle) &= \varepsilon^m \sum_{i_1=1}^{r_1} \dots \sum_{i_{N+1}=1}^{r_{N+1}} \sum_{(k_1, \dots, k_N) \in V_N^N} \pi_0(i_1, \dots, i_{N+1} : 1; k_2, \dots, k_N) \\ &\times \frac{\mu_{k_2}(i_{N+1})}{\lambda_{k_1}(i_{k_1})} \times \frac{\mu_{k_3}(i_{N+1})}{\lambda_{k_1}(i_{k_1}) + \lambda_{k_2}(i_{k_2})} \times \dots \times \frac{\mu_{k_{m+1}}(i_{N+1})}{\lambda_{k_1}(i_{k_1}) + \dots + \lambda_{k_m}(i_{k_m})} (1 + o(1)). \end{aligned}$$

Taking into account the exponentiality of $\tau_\varepsilon(i_1, \dots, i_{N+1} : s; k_1, \dots, k_{N-s})$ for fixed Θ we have

$$\begin{aligned} \mathbb{E} \exp\{i\varepsilon^m \Theta \tau_\varepsilon(i_1, \dots, i_{N+1} : 0; k_1, \dots, k_{N-s})\} \\ = 1 + \varepsilon^m \frac{i\Theta}{a_{i_1 i_1}^{(1)} + \dots + a_{i_{N+1} i_{N+1}}^{(N+1)} + \mu_{k_1}(i_{N+1})} (1 + o(1)), \end{aligned}$$

Notice that $\beta_\varepsilon = \varepsilon^m$ and therefore from Corollary 1 we immediately get that $\varepsilon^m \Omega_\varepsilon(m)$ converges weakly to an exponentially distributed random variable with parameter

$$\begin{aligned} \Lambda &= \sum_{i_1=1}^{r_1} \dots \sum_{i_{N+1}=1}^{r_{N+1}} \sum_{(k_1, \dots, k_N) \in V_N^N} \pi_0(i_1, \dots, i_{N+1} : 1; k_2, \dots, k_N) \\ &\times \frac{\mu_{k_2}(i_{N+1})}{\lambda_{k_1}(i_{k_1})} \times \frac{\mu_{k_3}(i_{N+1})}{\lambda_{k_1}(i_{k_1}) + \lambda_{k_2}(i_{k_2})} \times \dots \times \frac{\mu_{k_{m+1}}(i_{N+1})}{\lambda_{k_1}(i_{k_1}) + \dots + \lambda_{k_m}(i_{k_m})} \frac{1}{D}. \end{aligned} \quad (5)$$

which completes the proof.

It should be noted that this asymptotic approach considerably reduces the calculations since to solve the system of equations (2), (3) is much easier than to get the stationary distribution of $(Z_\varepsilon(t), t \geq 0)$ and system of stochastic relations concerning to its sojourn time in $\langle \alpha_m \rangle$. Notice, if $\mu_p(i_{N+1}) = \mu(i_{N+1})$, $p = 1, \dots, N$, a closed form solution can be obtained. Namely, by substituting (3) into (2) we get

$$\begin{aligned}
\pi_0(j_1, j_2, \dots, j_{N+1} : 0; k_1, \dots, k_N) &= \sum_{i_1 \neq j_1} \pi_0(i_1, j_2, \dots, j_{N+1} : 0; k_1, \dots, k_N) \\
&\times a_{i_1 j_1}^{(1)} \left(a_{i_1 i_1}^{(1)} + a_{j_2 j_2}^{(2)} + \dots + a_{j_{N+1} j_{N+1}}^{(N+1)} + \mu(j_{N+1}) \right)^{-1} \\
&+ \sum_{i_2 \neq j_2} \pi_0(j_1, j_2, \dots, j_{N+1} : 0; k_1, \dots, k_N) \\
&\times a_{i_2 j_2}^{(2)} \left(a_{j_1 j_1}^{(1)} + a_{i_2 i_2}^{(2)} + \dots + a_{j_{N+1} j_{N+1}}^{(N+1)} + \mu(j_{N+1}) \right)^{-1} + \dots \\
&+ \sum_{i_{N+1} \neq j_{N+1}} \pi_0(j_1, j_2, \dots, i_{N+1} : 0; k_1, \dots, k_N) \\
&\times a_{i_{N+1} j_{N+1}}^{(N+1)} \left(a_{j_1 j_1}^{(1)} + a_{j_2 j_2}^{(2)} + \dots + a_{i_{N+1} i_{N+1}}^{(N+1)} + \mu(i_{N+1}) \right)^{-1} \\
&+ \pi_0(j_1, \dots, j_{N+1} : 0; k_N, k_1, \dots, k_{N-1}) \mu(j_{N+1}) \left(a_{j_1 j_1}^{(1)} + \dots + a_{j_{N+1} j_{N+1}}^{(N+1)} + \mu(j_{N+1}) \right)^{-1}
\end{aligned} \tag{6}$$

Denote by $(\pi_{i_p}^{(p)}, i_p = 1, \dots, r_p)$ the steady-state distribution of the governing Markov chains $(\xi_p(t), t \geq 0)$, $p = 1, \dots, N + 1$, respectively. Clearly

$$\pi_{j_p}^{(p)} a_{j_p j_p}^{(p)} = \sum_{i_p \neq j_p} \pi_{i_p}^{(p)} a_{i_p j_p}^{(p)}, \quad p = 1, \dots, N + 1. \tag{7}$$

It can be verified, that the solution of (6) with (7) is

$$\pi_0(j_1, j_2, \dots, j_{N+1} : 0; k_1, \dots, k_N) = B \left(\pi_{j_1}^{(1)} \dots \pi_{j_{N+1}}^{(N+1)} \right) \left(a_{j_1 j_1}^{(1)} + \dots + a_{j_{N+1} j_{N+1}}^{(N+1)} + \mu(j_{N+1}) \right) \tag{8}$$

$$\pi_0(j_1, \dots, j_{N+1} : 1; k_1, \dots, k_{N-1}) = B \left(\pi_{j_1}^{(1)} \dots \pi_{j_{N+1}}^{(N+1)} \right) \mu(j_{N+1}) \tag{9}$$

where B is the normalizing constant, i.e.

$$\frac{1}{B} = N! \sum_{i_1=1}^{r_1} \dots \sum_{i_{N+1}=1}^{r_{N+1}} \left(\pi_{i_1}^{(1)} \dots \pi_{i_{N+1}}^{(N+1)} \right) \left(a_{i_1 i_1}^{(1)} + \dots + a_{i_{N+1} i_{N+1}}^{(N+1)} + 2\mu(i_{N+1}) \right).$$

Thus from (5) it follows that $\varepsilon^m \Omega(m)$ converges weakly to an exponentially distributed random variable with parameter

$$\begin{aligned}
\Lambda &= \frac{1}{N!} \sum_{i_1=1}^{r_1} \dots \sum_{i_{N+1}=1}^{r_{N+1}} \sum_{(k_1, \dots, k_N) \in V_N^N} \left(\pi_{i_1}^{(1)} \dots \pi_{i_{N+1}}^{(N+1)} \right) \mu(i_{N+1}) \\
&\times \frac{\mu(i_{N+1})}{\lambda_{k_1}(i_{k_1})} \times \frac{\mu(i_{N+1})}{\lambda_{k_1}(i_{k_1}) + \lambda_{k_2}(i_{k_2})} \times \dots \times \frac{\mu(i_{N+1})}{\lambda_{k_1}(i_{k_1}) + \dots + \lambda_{k_m}(i_{k_m})}.
\end{aligned} \tag{10}$$

Consequently, the distribution of the time while the number of active sources reaches the $(m+1)$ -th level for the first time is approximated by

$$\mathbb{P}(\Omega_\varepsilon(m) > t) = \mathbb{P}(\varepsilon^m \Omega_\varepsilon(m) > \varepsilon^m t) \approx \exp(-\varepsilon^m \Lambda t).$$

In particular, when $m=N-1$, we get that the busy period length of the receiver is asymptotically an exponentially distributed random variable with parameter

$$\begin{aligned} \varepsilon^{N-1}\Lambda &= \varepsilon^{N-1} \frac{1}{N!} \sum_{i_1=1}^{r_1} \dots \sum_{i_{N+1}=1}^{r_{N+1}} \sum_{(k_1, \dots, k_N) \in V_N^N} \left(\pi_{i_1}^{(1)} \dots \pi_{i_{N+1}}^{(N+1)} \right) \mu(i_{N+1}) \\ &\times \frac{\mu(i_{N+1})}{\lambda_{k_1}(i_{k_1})} \times \frac{\mu(i_{N+1})}{\lambda_{k_1}(i_{k_1}) + \lambda_{k_2}(i_{k_2})} \times \dots \times \frac{\mu(i_{N+1})}{\lambda_{k_1}(i_{k_1}) + \dots + \lambda_{k_{N-1}}(i_{k_{N-1}})}. \end{aligned} \quad (11)$$

In the case when there are no random environments, that is $\mu(i_{N+1}) = \mu$, $\lambda_p(i_p) = \lambda_p$, $i_p = 1, \dots, r_p$, $p = 1, \dots, N$, from (11) we get

$$\varepsilon^{N-1}\Lambda = \varepsilon^{N-1} \frac{1}{N!} \sum_{(k_1, \dots, k_N) \in V_N^N} \mu \frac{\mu}{\lambda_{k_1}} \frac{\mu}{\lambda_{k_1} + \lambda_{k_2}} \times \dots \times \frac{\mu}{\lambda_{k_1} + \dots + \lambda_{k_{N-1}}}. \quad (12)$$

Finally, for the totally homogeneous case from (12) we obtain

$$\varepsilon^{N-1}\Lambda = \frac{1}{(N-1)!} \frac{\mu^N}{(\lambda/\varepsilon)^{N-1}}. \quad (13)$$

The utilization U_r of the receiver, which is the long-run fraction of time during which it is busy, can be given by

$$U_r = \frac{1}{\varepsilon^{N-1}\Lambda} \left(\frac{1}{\varepsilon^{N-1}\Lambda} + I \right)^{-1}, \quad (14)$$

where I denotes its idle period length.

4 Numerical Results

This section presents a number of validation experiments (c.f., Tables 1-8) examining the credibility of the proposed approximation against exact results for the utilization at equilibrium. Note that, to best of our knowledge, an exact formula for it is known only for homogeneous systems which are not effected by random environments and it is given (via Palm-formula) by

$$U_r^* = \frac{\sum_{k=1}^N \binom{N}{k} k! \rho^k}{\sum_{k=1}^N \binom{N}{k} k! \rho^k},$$

where $\rho = \frac{\lambda/\varepsilon}{\mu}$. This is the reason why there are validation experiments for this case only. Instead of the overall utilization U_r , U_r^* let us consider the utilization of the receiver with respect to the p -th source, denoted by U_p, U_p^* , respectively. Clearly $U_p = U_r/N$, $U_p^* = U_r^*/N$, that is they are so-called normalized utilizations. In this case relations (13,14) reduce to the following approximation

$$U_p = \frac{1}{N} \frac{N!}{N! + \left(\frac{\mu}{\lambda/\varepsilon} \right)^N}.$$

The numerical results can be seen in Tables 1-8. It can be observed that the approximate values for $\{U_p\}$ are very much comperable in accuracy to those provided by the exact results for $\{U_p^*\}$. However, the computational complexity, due to the proposed approximation, has been considerably reduced. As λ/ε becomes greater than μ , the $\{U_p\}$ approximations, as expected, approach the exact values of $\{U_p\}$. Clearly, the greater the number of sources the less number of steps are needed to reach the exact results.

Table 1. $N = 3$			Table 2. $N = 4$		
ρ	U_p^*	U_p	ρ	U_p^*	U_p
1	0.3125	0.285714286	1	0.246153846	0.24
2	0.329113924	0.326530612	2	0.249605055	0.249350649
2^2	0.332657201	0.332467532	2^2	0.249968310	0.249959317
2^3	0.333237575	0.333224862	2^3	0.249997756	0.249997457
2^4	0.333320592	0.333319771	2^4	0.249999999	0.249999999
2^5	0.333333169	0.333331638	2^5	0.25	0.25
2^6	0.333333125	0.333333121			
2^7	0.333333307	0.333333307			
2^8	0.333333333	0.333333333			

Table 3. $N = 5$			Table 4. $N = 6$		
ρ	U_p^*	U_p	ρ	U_p^*	U_p
1	0.199386503	0.198347107	1	0.166581502	0.166435506
2	0.199968409	0.199947930	2	0.166664473	0.166666305
2^2	0.199998732	0.199998372	2^2	0.166666623	0.166666661
2^3	0.199999955	0.199999949	2^3	0.166666666	0.166666666
2^4	0.199999998	0.199999998			
2^5	0.2	0.2			

Table 5. $N = 7$			Table 6. $N = 8$		
ρ	U_p^*	U_p	ρ	U_p^*	U_p
1	0.142846715	0.142828804	1	0.124998860	0.1249969
2	0.142857009	0.142856921	2	0.124999993	0.124999988
2^2	0.142857142	0.142857141	2^2	0.125	0.125
2^3	0.142857143	0.142857143			

Table 7. $N = 9$			Table 8. $N = 10$		
ρ	U_p^*	U_p	ρ	U_p^*	U_p
1	0.111110998	0.111110805	1	0.099999999	0.099999999
2	0.111111111	0.111111111	2	0.1	0.1

5 Concluding Remarks

In this paper we have presented a queueing model to analyse the behaviour of a FCFS heterogeneous finite-source communication system with a single processor. The system operates in Markovian environments and the messages arrive fast compared to their service. An asymptotic approach has been provided to obtain the distribution of the time to the first system failure. The credibility of the method is illustrated with some simple validation experiments and favourable comparisons against exact results are made.

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