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ASYMPTOTIC ANALYSIS OF A HETEROGENEOUS RENEWABLE COMPLEX SYSTEM WITH RANDOM ENVIRONMENTS*

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Abstract

This paper deals with an asymptotic analysis of a complex renewable system with N heterogeneous elements looked after by n repairmen. Each element and the repair facility are assumed to operate in independent random environments governed by ergodic Markov chains. The operating and repair times of an element are supposed to be exponentially distributed random variables with parameter depending on the index of the element, the state of the corresponding random environment, and the indices of the failed elements. The repair is carried out according to a First Come, First Served (FCFS) discipline. Assuming that the repair rates are many times greater than the corresponding failure rates ("fast" repair), it is shown that the time to the first system failure converges in distribution, under appropriate norming, to an exponentially distributed random variable. Some numerical examples illustrate the effectiveness of the method proposed by comparing the approximate characteristics to the exact ones.

1. Introduction

Reliability analysis of renewable complex systems has been considered by many authors over the last years. They have used a variety of approaches and have made different assumptions about the statistical distributions of the operating and repair times. For an extensive bibliography on the basic models, reference may be made to Anisimov and Sztrik(1989), Franken *et al.*(1984), Gertsbakh(1984,1989), Rukhin and Hsieh(1987), Sztrik(1989). Systems with renewal of failed components usually have the characteristic that the average repair times are many times shorter than the average lifetimes. This "fast" repair property enables us to use asymptotic methods to determine the distribution of the first failure time of the system. It is also well-known that the great majority of problems can be treated by the help of Semi-Markov Processes (SMP). Since the failure-free operation time of the system corresponds to sojourn time problems we can use the results obtained for SMP. It is easy to see that in the case of "fast" repair the exit from a given

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subset of the underlying SMP is a "rare" event, that is, it occurs with a small probability. Thus, it is natural to investigate the asymptotic behaviour of sojourn time in a given subset, provided that the probability of exit from it tends to zero, see Anisimov *et al*(1987), Keilson(1979), Ushakov(1985). Refinements in the model are often needed when the the system environment is subject to randomly occuring fluctuations which appear as changes in the parameters of the model. These fluctuations may be due to the weather, earthquakes, or other changes in the physical environment, to personnel changes, to alteration of system usage intensity, etc., see Gaver *et al*.(1984), Sengupta(1990).

This paper deals with an asymptotic analysis of a complex renewable system with N heterogeneous elements looked after by n repairmen. Each element and the repair facility are assumed to operate in independent random environments governed by ergodic Markov chains. The operating and repair times of an element are supposed to be exponentially distributed random variables with parameter depending on the index of the element, the state of the corresponding random environment, and the indices of the failed elements. The repair is carried out according to a First Come, First Served (FCFS) discipline. Assuming that the repair rates are many times greater than the corresponding failure rates ("fast" repair), it is shown that the time to the first system failure converges in distribution, under appropriate norming, to an exponentially distributed random variable. Some numerical examples illustrate the effectiveness of the method proposed by comparing the approximate characteristics to the exact ones.

2. Preliminary results

In this section a brief survey is given of the most related theoretical results, mainly due to Anisimov, to be applied later on.

Let ($X_\varepsilon(k)$, $k\geq 0$) be a Markov chain with state space

$$\bigcup_{q=0}^{m+1} X_q, \quad X_i \cap X_j = 0, \ i\neq j,$$

defined by the transition matrix $(p_\varepsilon(i^{(q)},j^{(z)}))$ satisfying the following conditions:

1. $p_\varepsilon(i^{(0)},j^{(0)}) \to p_0(i^{(0)},j^{(0)})$, as $\varepsilon \to 0$, $i^{(0)},j^{(0)}\in X_0$, and matrix $P_0= (p_0(i^{(0)},j^{(0)}))$ is irreducible;

2. $p_\varepsilon(i^{(q)},j^{(q+1)}) = \varepsilon\alpha^{(q)}(i^{(q)},j^{(q+1)}) + o(\varepsilon)$, $i^{(q)}\in X_q$, $j^{(q+1)}\in X_{q+1}$;

3. $p_\varepsilon(i^{(q)},f^{(q)}) \to 0$, as $\varepsilon \to 0$, $i^{(q)},f^{(q)}\in X_q$, $q\geq 1$;

4. $p_\varepsilon(i^{(q)},f^{(z)}) \equiv 0$ $i^{(q)}\in X_q$, $f^{(z)}\in X_z$, $z-f\geq 2$.

In the sequel the set of states X_q is called the q-th level of the chain, q=1,...,m+1. Let us single out the subset of states

$$\langle\alpha_m\rangle = \bigcup_{q=0}^{m} X_q.$$

Denote by $\{\pi_\varepsilon(i^{(q)}),\ i^{(q)}\in X_q\}$, $q=1,\dots,m$ the stationary distribution of a chain with transition matrix

$$\left(\frac{p_\varepsilon(i^{(q)},j^{(z)})}{1-\sum\limits_{k^{(m+1)}\in X_{m+1}} p_\varepsilon(i^{(q)},k^{(m+1)})}\right),\quad i^{(q)}\in X_q,\ j^{(z)}\in X_z,\ q,z\le m.$$

Furthermore denote by $g_\varepsilon(<\alpha_m>)$ the steady state probability of exit from $<\alpha_m>$, that is

$$g_\varepsilon(<\alpha_m>) = \sum_{i^{(m)}\in X_m} \pi_\varepsilon(i^{(m)}) \sum_{j^{(m+1)}\in X_{m+1}} p_\varepsilon(i^{(m)},j^{(m+1)}).$$

Denote by $\{\pi_0(i^{(0)}),\ i^{(0)}\in X_0\}$ the stationary distribution corresponding to P_0 and let

$$\bar\pi_0 = \{\pi_0(i^{(0)}),\ i^{(0)}\in X_0\},\qquad \bar\pi_\varepsilon^{(q)} = \{\pi_\varepsilon(i^{(q)}), i^{(q)}\in X_q\}$$

be row vectors. Finally, let the matrix

$$A^{(q)} = (\alpha^{(q)}(i^{(q)},j^{(q+1)})),\quad i^{(q)}\in X_q,\ j^{(q+1)}\in X_{q+1},\ q=0,\dots,m$$

defined by condition 2.

Conditions (1)-(4) enable us to compute the main terms of the asymptotic expression for $\bar\pi_\varepsilon^{(q)}$ and $g_\varepsilon(<\alpha_m>)$. Namely, we obtain

$$\begin{aligned}\bar\pi_\varepsilon^{(q)} &= \varepsilon^q\bar\pi_0 A^{(0)}A^{(1)}\dots A^{(q-1)} + o(\varepsilon^q),\quad q=1,\dots,m,\\ g_\varepsilon(<\alpha_m>) &= \varepsilon^{m+1}\bar\pi_0 A^{(0)}A^{(1)}\dots A^{(m)}\underline{1} + o(\varepsilon^{m+1}),\end{aligned}\tag{1}$$

where $\underline{1}=(1,\dots,1)$ is a column vector, see Anisimov et al.(1987) pp. 141-153.

Let ($\eta_\varepsilon(t)$, $t\ge 0$) be a SMP given by the embedded Markov chain ($X_\varepsilon(k)$, $k\ge 0$) satisfying conditions (1)-(4). Let the times $\tau_\varepsilon(j^{(s)},k^{(z)})$- transition times from state $j^{(s)}$ to state $k^{(z)}$- fulfil the condition

$$E\exp\{i\theta\beta_\varepsilon\tau_\varepsilon(j^{(s)},k^{(z)})\} = 1+a_{jk}(s,z,\theta)\varepsilon^{m+1}+o(\varepsilon^{m+1}),\quad (i^2=-1)$$

where β_ε is some normalizing factor. Denote by $\Omega_\varepsilon(m)$ the instant at which the SMP reaches the (m+1)-th level for the first time, exit time from $<\alpha_m>$, provided $\eta_\varepsilon(0)\in<\alpha_m>$. Then we have:

Theorem 1. (cf. Anisimov *et al.*(1987) pp. 153) If the above conditions are satisfied then

$$\lim_{\varepsilon\to 0} E\exp\{i\theta\beta_\varepsilon\Omega_\varepsilon(m)\} = (1-A(\theta))^{-1}$$

where

$$A(\theta) = \frac{\sum\limits_{j^{(0)},k^{(0)}\in X_0} \pi_0(j^{(0)})p_0(j^{(0)},k^{(0)})a_{jk}(0,0,\theta)}{\bar\pi_0 A^{(0)}A^{(1)}\dots A^{(m)}\underline{1}}.$$

Corollary 1. In particular, if $a_{jk}(s,z,\theta)=i\theta m_{jk}(s,z)$ then the limit is an exponentially distributed random variable with mean

$$\frac{\sum\limits_{j^{(0)},k^{(0)}\in X_0} \pi_0(j^{(0)})p_0(j^{(0)},k^{(0)})m_{jk}(0,0)}{\bar{\pi}_0 A^{(0)}A^{(1)}\ldots A^{(m)}\underline{1}}.$$

3. The Queueing Model

Let us consider a renewable complex system consisting of N heterogeneous elements which are looked after by n operatives of the same kind. Element p is assumed to operate in a random environment governed by an ergodic Markov chain $(\xi_p(t),\ t\ge 0)$ with state space $(1,\ldots,r_p)$ and with transition density matrix $(a^{(p)}_{i_p j_p},\ i_p,j_p=1,\ldots,r_p,\ a^{(p)}_{i_p i_p}=\sum_{k\ne i_p} a^{(p)}_{i_p k})$. Whenever the environmental process $\xi_p(t)$ is in state i_p, there are s elements failed with indices $k_1,\ldots,k_s$, $s=0,\ldots,N-1$, the probability that element p breaks down in the time interval $(t,t+h)$ is $\lambda_p(i_p:k_1,\ldots,k_s)h+o(h)$, $p\in\{1,\ldots,N\}\setminus\{k_1,\ldots,k_s\}$. Each element is immediately repaired if there is an idle operative, otherwise a queueing line is formed. The service discipline is First Come, First Served (FCFS). The repair facility is also supposed to operate in a random environment governed by an ergodic Markov chain $(\xi_{N+1}(t),\ t\ge 0)$ with state space $(1,\ldots,r_{N+1})$ and with transition density matrix $(a^{(N+1)}_{i_{N+1}j_{N+1}},\ i_{N+1},j_{N+1}=1,\ldots,r_{N+1},\ a^{(N+1)}_{i_{N+1}i_{N+1}}=\sum_{k\ne i_{N+1}} a^{(N+1)}_{i_{N+1}k})$. Whenever the environmental process $\xi_{N+1}(t)$ is in state i_{N+1} and there are s elements stopped, with indices $k_1,\ldots,k_s$, $s=1,\ldots,N$, the probability that the repair of element p is completed in time interval $(t,t+h)$ is $\mu_p(i_{N+1}:k_1,\ldots,k_s;\varepsilon)h+o(h)$, $p\in\{k'_1,\ldots,k'_{\min(s,n)}\}$, where $\{k'_1,\ldots,k'_{\min(s,n)}\}$ denotes the indices of elements under repair. After being repaired each element immediately starts operating. All random variables involved here and the random environments are supposed to be independent of each other.

Let us consider the system under the assumption of "fast" repair, i.e., $\mu_p(i_{N+1}:k_1,\ldots,k_s;\varepsilon)\to\infty$ as $\varepsilon\to 0$.

For simplicity let $\mu_p(i_{N+1}:k_1,\ldots,k_s;\varepsilon)=\mu_p(i_{N+1}:k_1,\ldots,k_s)/\varepsilon$.

Denote by $Y_\varepsilon(t)$ the number of stopped elements at time t, and let

$$\Omega_\varepsilon(m)=\inf\{t:\ t>0,\ Y_\varepsilon(t)=m+1\ /\ Y_\varepsilon(0)\le m\},$$

that is, the instant at which the number of stopped elements reaches the (m+1)-th level for the first time, provided that at the beginning their number is not greater than m; $m=1,\ldots,N-1$. In the following $\Omega_\varepsilon(m)$ is referred to as the first system failure time.

Denote by $(\pi^{(p)}_{i_p},\ i_p=1,\ldots,r_p)$, $p=1,\ldots,N+1$ the steady-state distribution of the governing Markov chains $(\xi_p(t),\ t\ge 0)$, respectively, and let V^s_N be the set of all variations of order s of integers $1,\ldots,N$.

Now we have:

Theorem 2. For the system in question under the above assumptions, independently of the initial state, the distribution of the normalized random variable $\varepsilon^m \Omega_\varepsilon(m)$ converges weakly to an exponentially distributed random variable with parameter

$$\Lambda = \sum_{i_1=1}^{r_1} \cdots \sum_{i_{N+1}=1}^{r_{N+1}} \sum_{(k_1,\dots,k_{m+1})\in V_N^{m+1}} \left(\prod_{p=1}^{N+1} \pi_{i_p}^{(p)} \right) \frac{\prod_{s=0}^{m} \lambda_{k_{s+1}}(i_{k_{s+1}} : k_1,\dots,k_s)}{\prod_{s=1}^{m} \sum_{q=1}^{\min(s,n)} \mu_{k_q}(i_{N+1} : k_1,\dots,k_s)} .$$

Proof. It is easy to see that the process

$$Z_\varepsilon(t) = (\xi_1(t),\dots,\xi_{N+1}(t) : Y_\varepsilon(t); \gamma_1(t),\dots,\gamma_{Y_\varepsilon(t)}(t))$$

is a multi-dimensional Markov chain with state space

$$E = ((i_1,\dots,i_{N+1} : s; k_1,\dots k_s),\quad i_p=1,\dots,r_p,\quad p=1,\dots,N+1,\quad (k_1,\dots k_s)\in V_N^s,\quad s=0,\dots,N),$$

where $\gamma_1(t),\dots,\gamma_{Y_\varepsilon(t)}(t)$ denote the indices of failed elements at time t in the order of their breakdowns, and by definition $k_o = \{0\}$.
Furthermore, let

$$<\alpha_m> = ((i_1,\dots i_{N+1} : s; k_1,\dots k_s),\quad i_p=1,\dots,r_p,\quad p=1,\dots,N+1,\quad (k_1,\dots k_s)\in V_N^s,\quad s=0,\dots,m).$$

Hence our aim is to determine the distribution of the first exit time of $Z_\varepsilon(t)$ from $<\alpha_m>$, provided that $Z_\varepsilon(o)\in <\alpha_m>$.
It can easily be verified that the transition probabilities in any time interval (t, t+h) are the following:

$$(i_1,\dots,i_{N+1} : s; k_1,\dots,k_s) \xrightarrow{h}$$

$$\begin{cases} (i_1,\dots,j_p,\dots,i_{N+1} : s; k_1,\dots,k_s) & a_{i_p j_p}^{(p)} h + o(h), \quad s=0,\dots,N, \\ & j_p \neq i_p, \quad p=1,\dots,N+1, \\ (i_1,\dots,i_{N+1} : s+1; k_1,\dots,k_{s+1}) & \lambda_{k_{s+1}}(i_{k_{s+1}} : k_1,\dots,k_s)h + o(h, \quad s=0,\dots,N-1, \\ (i_1,\dots,i_{N+1} : s-1; k_1,\dots,k_{q-1},k_{q+1},\dots,k_s) & \mu_{k_q}(i_{N+1} : k_1,\dots,k_s)h/\varepsilon + o(h), \\ & s=1,\dots,N, \quad q=1,\dots,\min(s,n). \end{cases}$$

In addition, the sojourn time $\tau_\varepsilon(i_1,\dots,i_{N+1} : s; k_1,\dots,k_s)$ of $Z_\varepsilon(t)$ in state $(i_1,\dots,i_{N+1} : s; k_1,\dots,k_s)$ is exponentially distributed with parameter

$$a_{i_1 i_1}^{(1)} + \dots + a_{i_{N+1} i_{N+1}}^{(N+1)} + \sum_{j\neq k_1,\dots,k_s}^{N} \lambda_j(i_j : k_1,\dots,k_s)$$

$$+\sum_{q=1}^{\min(s,n)} \mu_{k_q}(i_{N+1}:k_1,\dots,k_s)/\varepsilon.$$

where by definition $\mu_{k_q}(i_{N+1}:0)=0.$

Thus, the transition probabilities for the embedded Markov chain are

$$p_\varepsilon[(i_1,\dots,i_{N+1}:s;k_1,\dots,k_s),(i_1,\dots,j_p,\dots,i_{N+1}:s;k_1,\dots,k_s)]$$

$$=\frac{a^{(p)}_{i_p j_p}}{a^{(1)}_{i_1 i_1}+\dots+a^{(N+1)}_{i_{N+1} i_{N+1}}+\sum_{j\neq k_1,\dots,k_s}\lambda_j(i_j:k_1,\dots,k_s)+\sum_{q=1}^{\min(s,n)}\mu_{k_q}(i_{N+1}:k_1,\dots,k_s)/\varepsilon},$$

for $s=0,\dots,N,\quad p=1,\dots,N+1,$

$$p_\varepsilon[(i_1,\dots,i_{N+1}:s;k_1,\dots,k_s),(i_1,\dots,i_{N+1}:s+1;k_1,\dots,k_{s+1})]$$

$$=\frac{\lambda_{k_{s+1}}(i_{k_{s+1}}:k_1,\dots,k_s)}{a^{(1)}_{i_1 i_1}+\dots+a^{(N+1)}_{i_{N+1} i_{N+1}}+\sum_{j\neq k_1,\dots,k_s}\lambda_j(i_j:k_1,\dots,k_s)+\sum_{q=1}^{\min(s,n)}\mu_{k_q}(i_{N+1}:k_1,\dots,k_s)/\varepsilon},$$

for $s=0,\dots,N-1,$

$$p_\varepsilon[(i_1,\dots,i_{N+1}:s;k_1,\dots,k_s),(i_1,\dots,i_{N+1}:s-1;k_1,\dots,k_{q-1},k_{q+1},\dots,k_s)]$$

$$=\frac{\mu_{k_q}(i_{N+1}:k_1,\dots,k_s)/\varepsilon}{a^{(1)}_{i_1 i_1}+\dots+a^{(N+1)}_{i_{N+1} i_{N+1}}+\sum_{j\neq k_1,\dots,k_s}\lambda_j(i_j:k_1,\dots,k_s)+\sum_{p=1}^{\min(s,n)}\mu_{k_p}(i_{N+1}:k_1,\dots,k_s)/\varepsilon},$$

for $q=1,\dots,\min(s,n),\quad s=1,\dots,N.$

As $\varepsilon\to 0$ this implies

$$p_\varepsilon[(i_1,\dots,i_{N+1}:0;0),(i_1,\dots,j_p,\dots,i_{N+1}:0;0)]$$

$$=\frac{a^{(p)}_{i_p j_p}}{a^{(1)}_{i_1 i_1}+\dots+a^{(N+1)}_{i_{N+1} i_{N+1}}+\sum_{j=1}^{N}\lambda_j(i_j:0)},\qquad p=1,\dots,N+1,$$

$$p_\varepsilon[(i_1,\dots,i_{N+1}:s;k_1,\dots,k_s),(i_1,\dots,j_p,\dots,i_{N+1}:s;k_1,\dots,k_s)] = o(1),\quad s>1$$

$$p_\varepsilon[(i_1,\dots,i_{N+1}:0;0),(i_1,\dots,i_{N+1}:1;k)]$$

$$= \frac{\lambda_k(i_k:0)}{a^{(1)}_{i_1 i_1} + \ldots + a^{(N+1)}_{i_{N+1} i_{N+1}} + \sum_{j=1}^{N} \lambda_j(i_j:0)},$$

$$p_\varepsilon[(i_1,\ldots,i_{N+1}:s;k_1,\ldots,k_s),(i_1,\ldots,i_{N+1}:s+1;k_1,\ldots,k_{s+1})]$$

$$= \frac{\lambda_{k_{s+1}}(\, i_{k_{s+1}}:k_1,\ldots,k_s)\varepsilon}{\sum_{q=1}^{\min(s,n)} \mu_{k_q}(i_{N+1}:k_1,\ldots,k_s)}(1+o(1)), \qquad s=1,\ldots,N-1.$$

This agrees with the conditions (1)-(4), but here the zero level is the set

$$((i_1,\ldots,i_{N+1}:0;0),\ (i_1,\ldots,i_{N+1}:1;k)\ \ i_p=1,\ldots,r_p,\ \ p=1,\ldots,N+1,\ \ k=1,\ldots,N),$$

while the q-th level is the set

$$((i_1,\ldots,i_{N+1}:q+1;k_1,\ldots,k_{q+1}),\ i_p=1,\ldots,r_p,\ p=1,\ldots,N+1,\ (k_1,\ldots,k_{q+1})\in V_N^{q+1}).$$

Since the level 0 in the limit forms an essential class, the probabilities $\pi_o(i_1,\ldots,i_{N+1}:0;0)$, $\pi_o(i_1,\ldots,i_{N+1}:1;k)$, $i_p=1,\ldots,r_p$, $p=1,\ldots,N+1$, $k=1,\ldots,N$ satisfy the following system of equations

$$\pi_o(j_1,\ldots,j_{N+1}:0;0) = \sum_{i_1\neq j_1} \pi_o(i_1,j_2,\ldots,j_{N+1}:0;0)$$

$$\times\, a^{(1)}_{i_1 j_1}\Big(a^{(1)}_{i_1 i_1} + a^{(2)}_{j_2 j_2} + \ldots + a^{(N+1)}_{j_{N+1} j_{N+1}} + \lambda_1(i_1:0) + \lambda_2(j_2:0) + \ldots + \lambda_N(j_N:0)\Big)^{-1}$$

$$+ \sum_{i_2\neq j_2} \pi_o(j_1,i_2,\ldots,j_{N+1}:0;0)$$

$$\times\, a^{(2)}_{i_2 j_2}\Big(a^{(1)}_{j_1 j_1} + a^{(2)}_{i_2 i_2} + \ldots + a^{(N+1)}_{j_{N+1} j_{N+1}} + \lambda_1(j_1:0) + \lambda_2(i_2:0) + \ldots + \lambda_N(j_N:0)\Big)^{-1} +$$

$$\ldots + \sum_{i_{N+1}\neq j_{N+1}} \pi_o(j_1,j_2,\ldots,i_{N+1}:0;0)$$

$$\times\, a^{(N+1)}_{i_{N+1} j_{N+1}}\Big(a^{(1)}_{j_1 j_1} + a^{(2)}_{j_2 j_2} + \ldots + a^{(N+1)}_{i_{N+1} i_{N+1}} + \lambda_1(j_1:0) + \lambda_2(j_2:0) + \ldots + \lambda_N(j_N:0)\Big)^{-1}$$

$$+ \pi_o(j_1,\ldots,j_{N+1}:1;k), \tag{2}$$

$$\pi_o(j_1,\ldots,j_{N+1}:1;k) = \pi_o(j_1,\ldots,j_{N+1}:0;0) \tag{3}$$

$$\times\, \lambda_k(j_k:0)\Big(a^{(1)}_{j_1 j_1} + a^{(2)}_{j_2 j_2} + \ldots + a^{(N+1)}_{j_{N+1} j_{N+1}} + \lambda_1(j_1:0) + \lambda_2(j_2:0) + \ldots + \lambda_N(j_N:0)\Big)^{-1}.$$

It is clear that

$$\pi^{(p)}_{j_p} a^{(p)}_{j_p j_p} = \sum_{i_p \neq j_p} \pi^{(1)}_{i_p} a^{(p)}_{i_p j_p}, \qquad p=1,\dots,N+1. \tag{4}$$

It can easily be verified, that the solution of (2),(3) with (4) is

$$\pi_o(i_1,\dots,i_{N+1}:0;0) = B\left(\pi^{(1)}_{i_1}\cdots\pi^{(N+1)}_{i_{N+1}}\right)\left(a^{(1)}_{i_1 i_1}+\dots+a^{(N+1)}_{i_{N+1} i_{N+1}} + \sum_{k=1}^{N}\lambda_k(i_k:0)\right),$$

$$\pi_o(i_1,\dots,i_{N+1}:1;k) = B\left(\pi^{(1)}_{i_1}\cdots\pi^{(N+1)}_{i_{N+1}}\right)\lambda_k(i_k:0),$$

where B is the normalizing constant, i.e.

$$1/B = \sum_{i_1=1}^{r_1}\cdots\sum_{i_{N+1}=1}^{r_{N+1}}\left(\pi^{(1)}_{i_1}\cdots\pi^{(N+1)}_{i_{N+1}}\right)\left(a^{(1)}_{i_1 i_1}+\dots+a^{(N+1)}_{i_{N+1} i_{N+1}} + 2\sum_{k=1}^{N}\lambda_k(i_k:0)\right).$$

By using formula (1) it is easy to show that the probability of exit from $<\alpha_m>$ is

$$B\,\varepsilon^m \sum_{i_1=1}^{r_1}\cdots\sum_{i_{N+1}=1}^{r_{N+1}}\sum_{(k_1,\dots,k_{m+1})\in V_N^{m+1}}\left(\prod_{p=1}^{N+1}\pi^{(p)}_{i_p}\right)\frac{\prod_{s=0}^{m}\lambda_{k_{s+1}}(i_{k_{s+1}}:k_1,\dots,k_s)}{\prod_{s=1}^{m}\sum_{q=1}^{\min(s,n)}\mu_{k_q}(i_{N+1}:k_1,\dots k_s)}$$

$$\times\ (1+o(1)).$$

Taking into account the exponentiality of $\tau_\varepsilon(j_1,\dots,j_{N+1}:s;k_1,\dots,k_s)$ for fixed θ we have

$$E\exp\{i\varepsilon^m\theta\tau_\varepsilon(j_1,\dots,j_{N+1}:0;0)\} =$$

$$= 1+\varepsilon^m\frac{i\theta}{a^{(1)}_{j_1 j_1}+\dots+a^{(N+1)}_{j_{N+1} j_{N+1}} + \sum_{k=1}^{N}\lambda_k(j_k:0)}(1+o(1)),$$

$$E\exp\{i\varepsilon^m\theta\tau_\varepsilon(j_1,\dots,j_{N+1}:s;k_1,\dots,k_s)\} = 1+o(\varepsilon^m), \qquad s>0.$$

Notice that $\beta_\varepsilon=\varepsilon^m$ and therefore from Corollary 1 we immediately get the statement that $\varepsilon^m\Omega_\varepsilon(m)$ converges weakly to an exponentially distributed random variable with parameter

$$\Lambda = \sum_{i_1=1}^{r_1}\cdots\sum_{i_{N+1}=1}^{r_{N+1}}\sum_{(k_1,\dots,k_{m+1})\in V_N^{m+1}}\left(\prod_{p=1}^{N+1}\pi^{(p)}_{i_p}\right)\frac{\prod_{s=0}^{m}\lambda_{k_{s+1}}(i_{k_{s+1}}:k_1,\dots,k_s)}{\prod_{s=1}^{m}\sum_{q=1}^{\min(s,n)}\mu_{k_q}(i_{N+1}:k_1,\dots,k_s)},$$

which completes the proof.

Consequently, the distribution of the failure-free operation time of the system, $\Omega_\varepsilon(m)$, can be approximated by

$$P(\Omega_\varepsilon(m)>t) = P(\varepsilon^m\Omega_\varepsilon(m)>\varepsilon^m t) \approx \exp(-\varepsilon^m\Lambda t),$$

i.e. $\Omega_\varepsilon(m)$ is asymptotically an exponentially distributed random variable with parameter $\varepsilon^m\Lambda$.

In particular, for $m=N-1$, which means that there is no operating element, we have

$$\varepsilon^{N-1}\Lambda=\varepsilon^{N-1}\sum_{i_1=1}^{r_1}\cdots\sum_{i_{N+1}=1}^{r_{N+1}}\sum_{(k_1,\dots,k_N)\in V_N^N}\left(\prod_{p=1}^{N+1}\pi_{i_p}^{(p)}\right)\frac{\prod_{s=0}^{N-1}\lambda_{k_{s+1}}(i_{k_{s+1}}:k_1,\dots,k_s)}{\prod_{s=1}^{N-1}\sum_{q=1}^{\min(s,n)}\mu_{k_q}(i_{N+1}:k_1,\dots,k_s)}.$$

In the case, where the failure and repair rates depend only on the number of failed elements, that is,

$$\lambda_p(i_p:k_1,\dots,k_s) = \lambda_p(i_p;s), \quad \mu_p(i_{N+1}:k_1,\dots,k_s) = \mu_p(i_{N+1};s),$$

we get

$$\varepsilon^{N-1}\Lambda = \varepsilon^{N-1}\sum_{i_1=1}^{r_1}\cdots\sum_{i_{N+1}=1}^{r_{N+1}}\sum_{(k_1,\dots,k_N)\in V_N^N}\left(\prod_{p=1}^{N+1}\pi_{i_p}^{(p)}\right)\frac{\prod_{s=0}^{N-1}\lambda_{k_{s+1}}(i_{k_{s+1}};s)}{\prod_{s=1}^{N-1}\sum_{q=1}^{\min(s,n)}\mu_{k_q}(i_{N+1};s)}. \qquad (5)$$

Furthermore, if there are no random environments from (5) we have

$$\Lambda^* = \varepsilon^{N-1}\Lambda = \varepsilon^{N-1}\sum_{(k_1,\dots,k_N)\in V_N^N}\frac{\prod_{s=0}^{N-1}\lambda_{k_{s+1}}(s)}{\prod_{s=1}^{N-1}\sum_{q=1}^{\min(s,n)}\mu_{k_q}(s)}, \qquad (6)$$

where $\lambda_p(i_p;s) = \lambda_p(s)$, $i_p=1,\dots,r_p$, $\mu_p(i_{N+1};s) = \mu_p(s)$, $i_{N+1}=1,\dots,r_{N+1}$.

Moreover, if each element has the same repair rate μ, and the failure rates do not depend on the number of stopped elements then (6) yields

$$\Lambda^* = \varepsilon^{N-1}\Lambda = \frac{N!}{n!\,n^{N-n-1}}\frac{\lambda_1\cdots\lambda_N}{(\mu/\varepsilon)^{N-1}}. \qquad (7)$$

Finally, for homogeneous failure rates from (7) we have

$$\Lambda^* = \varepsilon^{N-1}\Lambda = \frac{N!}{n!\,n^{N-n-1}}\frac{\lambda^N}{(\mu/\varepsilon)^{N-1}}. \qquad (8)$$

Hence, by using (5) the steady-state probability, Q_W, that at least one element works is

$$Q_W = \frac{1}{\varepsilon^{N-1}\Lambda}\left(\frac{1}{\varepsilon^{N-1}\Lambda} + B_N\right)^{-1},$$

where B_N denotes the mean period of time during which all elements are stopped. It is quite easy to verify that from (7) we obtain

$$Q_W = \left(1 + \frac{N!}{n!\,n^{N-n}} \frac{\lambda_1 \cdots \lambda_N}{(\mu/\varepsilon)^N} \right)^{-1} . \tag{9}$$

Finally, for the simplest case we have

$$Q_W = 1 \,/\, \left(1 + \frac{N!}{n!\,n^{N-n}} \left(\frac{\lambda}{\mu/\varepsilon} \right)^N \right) \tag{10}$$

4. Some Numerical Results

In this section some numerical examples are given to illustrate the effectiveness of the method proposed by comparing the approximate results to the exact ones.

Case 1.

Here $\rho = \frac{\lambda}{\mu/\varepsilon}$ and the exact steady-state probability, P_W, that at least one element works is $P_W = 1 - \frac{N!}{n!\,n^{N-n}} \rho^N P_o$ (from Palm-formula).

With $n = 3$ by using (8), (10) we get the following results:

N = 5

ρ	P_W	Q_W	$1/\Lambda^*$
1	0.936305732	0.310344828	0.1
2^{-1}	0.991023339	0.935064935	2.4
2^{-2}	0.999290680	0.997834560	38.4
2^{-3}	0.999962376	0.999932188	614.4
2^{-4}	0.999998435	0.999997881	9830.4
2^{-5}	0.999999943	0.999999934	157286.4
2^{-6}	0.999999998	0.999999998	2516582.4
2^{-7}	1	1	40265318.4

N = 15

ρ	P_W	Q_W	$1/\Lambda^*$
1	0.950212859	2.43840 E-6	8.1280 E-7
2^{-1}	0.997516417	7.39898 E-2	1.3316 E-2
2^{-2}	0.999991894	0.999618207	218.1
2^{-3}	0.999999998	0.999999988	3574746.5
2^{-4}	1	1	5.856864 E10

N = 25

ρ	P_W	Q_W	$1/\Lambda^*$
1	0.950212932	1.2138 E-14	4.0462 E-15
2^{-1}	0.997521248	4.07307 E-7	6.78846 E-8
2^{-2}	0.999993850	0.93181969	1.1
2^{-3}	1	0.99999999	19107840.5
2^{-4}	1	1	3.20576 E20

We can see how Q_W depends on N, ρ and how accurate it is. It should be noted that the greater the N the less the ρ for an acceptable approximation. Furthermore, we can observe the sharp increase in $1/\Lambda^*$.

Case 2.

In this section we deal with heterogeneous failures and homogeneous repair rates. By using (7), (9) we get the asymptotic results against P_W derived by Sztrik(1985). The input parameters are the following:

$$N = 4, \quad n = 2, \quad \lambda_1 = 1, \ \lambda_2 = 2, \ \lambda_3 = 3, \ \lambda_4 = 4,$$

We have:

μ/ε	P_W	Q_W	$1/\Lambda^*$
1	0.626943005	1.36986 E-2	6.94444 E-3
5	0.977478886	0.896700144	0.8
10	0.997039717	0.992851469	6.9
20	0.999718279	0.999550202	55.5
30	0.999935358	0.999911119	187.5

We can see how the characteristics depend on the repair rate.

5. Conclusion

In this paper a renewable complex system has been treated assuming that each element and the repair facility operate in independent random environment governed by ergodic Markov chains. The life and repair times of an element are supposed to be exponentially distributed random variables with parameter depending on the index of the element, the indices of the stopped components, and state of the corresponding random environment. Assuming that the repair rates are many times greater than the corresponding failure rates ("fast" repair), it is shown that the failure-free operation time of the system converges weakly, under appropriate norming, to an exponentially distributed random variable. Furthermore, some numerical examples illustrate the problem in question by comparing the approximate characteristics to the exact ones.

References

Anisimov V.V., Zakusilo O.K., Donchenko V.S. (1987) *Elements of queueing theory and asymptotic analysis of systems,* Visha Skola, Kiev (in Russian).

Anisimov V.V., Sztrik J. (1989) Asymptotic analysis of some controlled finite-source queueing systems, *Acta Cybernetica* 9 27-38.

Franken P., Kirstein B.M., Streller A. (1984) Reliability Analysis of Complex Systems with Repair, *EIK* **20** 407-422.

Gaver D.P., Jacobs P.A., Latouche G. (1984) Finite birth-and-death models in randomly changing environments, *Advances in Applied Probability* **16** 715-731.

Gertsbakh I.B. (1984) Asymptotic methods in reliability theory; A review, *Advances in Applied Probability* **16** 157-175.

Gertsbakh I.B. (1989) *Statistical Reliability Theory,* Marcel Dekker, Inc. New York.

Keilson J. (1979) *Markov chain models - Rarity and exponentiality,* Springer-Verlag, Berlin.

Rukhin A.L., Hsieh H.K. (1987) Survey of Soviet Works on Reliability Theory, *Statistical Science* 2 284-503.

Sengupta B. (1990) A queue with service interruptions in an alternating random environment, *Operations Ressearch* **38** 308-318.

Sztrik J. (1985) On the finite-source $\vec{G}/M/r$ queues, *European Journal of Operational Research* **20** 216-268.

Sztrik J. (1989) Asymptotic Reliability Analysis of Some Complex Systems with Repair Operating in Random Environments, *J. Inf. Process. Cybern. EIK* **25** 37-43.

Ushakov I.A. (1985) (Editor) Reliability of Complex Systems; Handbook, *Radio i Svjaz,* Moscow (in Russian).