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Asymptotic waiting time analysis of finite source $M/GI/1$ retrial queueing systems with conflicts and unreliable server

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Abstract. The goal of the present paper is to analyze the steady-state distribution of the waiting time in a finite source $M/G/1$ retrial queueing system where conflicts may happen and the server is unreliable. An asymptotic method is used when the number of source N tends to infinity, the arrival intensity from the sources, the intensity of repeated calls tend to zero, while service intensity, breakdown intensity, recovery intensity are fixed. It is proved that the limiting steady-state probability distribution of the number of transitions/retrials of a customer into the orbit is geometric, and the waiting time of a customer is generalized exponentially distributed. The average total service time of a customer is also determined. Our new contribution to this topic is the inclusion of breakdown and recovery of the server. Prelimit distributions obtained by means of stochastic simulation are compared to the asymptotic ones and several numerical examples illustrate the power of the proposed asymptotic approach.

1. Introduction

Finite source queueing systems with repeated attempts are extremely popular and powerful stochastic models to evaluate the performance of complex telecommunication networks, call centers, sensor networks, wireless communication systems, etc. For an overview on this topic, which is a modification of the machine interference problem (see [16], [42]), we can refer to, for example, [1], which deals

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with modeling of call centers; [4] is the well-known book on retrial queueing systems; [8] treats finite-source retrial systems with two-way communication; [12] is an important paper on $M/G/1//N$ retrial queue; [13] is the classical book on retrial systems; [15] deals with related models of retrial queues; [19] is a survey of retrial queueing systems; and [27] investigates finite-source systems with vacation.

Since in practice the server may break down, i.e., it is unreliable and after a failure it needs a repair, it is very important to investigate this type of systems. Retrial queues with a finite number of sources when the server is unreliable have been considered in, for example [3], where the involved random variables are exponentially distributed and there is no collision; [10] investigates with the distribution of the retrials without collision; [14] uses an algorithmic approach to get the distribution of the number of customers in the orbit when all the involved random variables are exponentially distributed, [17] applies Markov-Renewal Stochastic Petri Net for the analysis; [39] investigates $M/M/1//N$ retrial systems with unreliable sources and a server without collision; [46], [47] and [50] deal with $M/G/1//N$ retrial systems with unreliable server without collision using the discrete-transformation method to get the steady-state distribution of the system state.

It should be noted that, surprisingly, only a few papers have dealt with systems when the arriving customers (primary or secondary) collide with the customer under service and both enter into the orbit, see, for example, [2], [7], [20], [21], [37]. Unfortunately, conflicts decrease the effectiveness of the system performance, and that is why new protocols should be developed. Without going into details and describing the protocols, we mention some patents where the collision is investigated, see, for example, [6], [18], [26], [48]. Hence, the mathematical modeling of such systems is of basic interest. Investigations of queueing systems with collision is important not only from a mathematical point of view, but from that of application fields, too. Some examples are the following: wireless communication systems for multiserver transmissions, random access communication systems, systems with unslotted CSMA/CD protocols.

Nazarov and his colleagues have developed a very powerful asymptotic method (see [36]) for the investigation of various queueing models. Finite source retrial systems with conflicts have recently been treated among others in the papers [22]–[25] and [28].

Sztrik and his colleagues have modeled systems with an unreliable server, see, for example, [3], [11], [38], [43], [44], [49], and that is why the two research groups combined their efforts in 2017.

The unique contribution of our new model is a natural modification of the $M/G/1//N$ retrial system treated in [23], where the server was reliable and the $M/M/1//N$ system with an unreliable server analyzed in [33]. It is the continuation of [35] where the asymptotic distribution of the number of customers in the system was investigated. In [45], the present model has been analyzed by means of stochastic simulation.

In this paper, an $M/GI/1//N$ retrial queueing system with conflicts of requests and a server subject to failure and repair is treated. Using asymptotic analysis when N tends to infinity, it is proved that the limiting probability distribution of the number of retrials is geometric with a given parameter. Based on this, the prelimit distribution of the waiting time of a customer is obtained, and by several examples, the power of the proposed approximations is illustrated.

When N tends to infinity, a natural question arises, why do not we use the results concerning systems with infinity source, that is, when the arrival process is Poisson? We have the following explanations in general:

- The performance measures should depend on the number of sources, denoted by N.
- Even in the case of classical $M/G/1$ and $M/G/1//N$ systems, the analysis of the steady-state distribution of the number of customers in the system is totally different. In the case of $M/G/1$, the steady state-distribution of the queue length distribution can be computed in a recursive way (see, for example, [5], [40]), but in the case of $M/G/1//N$, the distribution can be obtained in a closed-form even if the formula is not simple.
- The calculation of the waiting time distribution is even more complicated, using exact methods, only the Laplace transform of the waiting time can be given in a rather complicated way.
- For retrial systems, the investigation is much more complicated. The results concern mainly systems without collision, only few papers deal with collisions, mainly because of the mathematical complications due to the non-neighboring transitions in the number of customers in the orbit.
- The authors are not aware of any paper dealing with $M/G/1$ retrial systems with collision and unreliable server where the distribution of the waiting time is given in a closed form.

Sections 2 deals with the description of the model. In Section 3, the elapsed service time as a supplementary variable is introduced and the corresponding Kolmogorov equations are obtained. Sections 4 and 5 are concerned with the distribution of the number of transitions of a customer carries until the successful

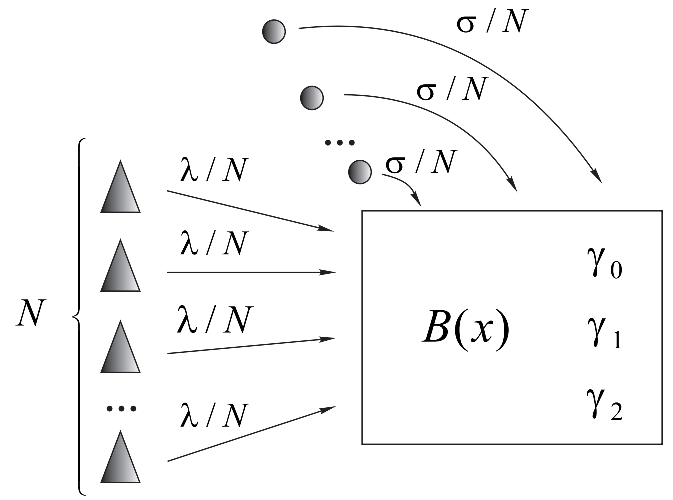

Figure 1. $M/GI/1//N$ retrial queueing system with conflicts and an unreliable server.

completion of its service. Section 6 is devoted to the distribution of the waiting time of a customer in prelimit situation. In Section 7, the limiting average total time of a customer under service is given. In Section 8, several numerical examples and comparisons to simulation results are considered, illustrating the power of the applied asymptotic method. Then several comments are made, and finally a Conclusion completes the paper.

2. Description of the model and notations

In Figure 1, the operation of an $M/GI/1//N$ retrial system with conflicts of customers and an unreliable server is described. We have N sources and each of them can send a primary call with intensity λ/N to the service facility. A source cannot generate a new customer until the end of the successful service of the current one. If a call finds the server idle, its service starts immediately.

Let $B(x)$ denote the distribution function of the required service time, $\mu(y) = B^{'}(y)(1-B(y))^{-1}$ its service rate function, and $B^{*}(y)$ its Laplace–Stieltjes transform, respectively. Otherwise, if the server is busy, an arriving (primary or repeated) call causes conflict with the request under service and they both move into the orbit. Times between the retrials are supposed to be exponentially distributed random variables with intensity σ/N. Assuming that the server is subject to breakdowns, the failure-free operation times are supposed to be exponentially distributed with failure intensity γ_0 if the server is idle, and with intensity γ_1 if it is busy. As soon as the server fails, the repair starts and the recovery times are assumed to be exponentially distributed with intensity γ_2. We consider the situation when the server is broken, all sources can generate requests and send them into the orbit. In addition, customers may repeat their calls from the orbit for service, but all arriving customers immediately return to the orbit. Finally, we assume that the interrupted customer is directed to the orbit and its next service is independent of the interrupted one. All random variables involved in the model construction are supposed to be independent of each other.

Let $Q(t)$ denote the number of requests in the system at time t, that is, the total number of requests in the orbit and in service. Similarly, let $C(t)$ denote the state of the server at time t, that is

$$C(t) = \begin{cases} 0 & \text{if the server is idle at time } t, \\ 1 & \text{if the server is busy at time } t, \\ 2 & \text{if the server is failed at time } t. \end{cases}$$

Hence, we will deal with process $\{C(t), Q(t)\}$, which is not Markovian. Therefore, as usual, we introduce the elapsed service time as a supplementary variable.

3. Kolmogorov equations for the probability distribution

Let $y(t)$ denote the supplementary random variable, equal to the elapsed service time of the customer under service till time t.

It is clear that $\{C(t), Q(t), y(t)\}$ is a Markov process. Let us note, that process $y(t)$ is defined only at $C(t) = 1$.

Let us define the steady-state probabilities and density function as follows:

$$p_0(j) = P\{C(t) = 0, Q(t) = j\},$$
$$p_1(j, y) = \frac{\partial P\{C(t) = 1, Q(t) = j, y(t) < y\}}{\partial y},$$
$$p_1(j) = P\{C(t)=1, Q(t)=j\} = \int_0^\infty p_1(j, y)dy, \quad p_2(j) = P\{C(t)=2, Q(t)=j\}.$$

As usual, for $p_0(j)$, $p_1(j, y)$ and $p_2(j)$, the following system of Kolmogorov equations can be obtained:

$$
\begin{aligned}
&-\left[\lambda\frac{N-j}{N}+\frac{j}{N}\sigma+\gamma_0\right]p_0(j)+\int\limits_0^\infty p_1(j+1,y)\mu(y)dy+\lambda\frac{N-j+1}{N}p_1(j-1)\\
&\quad+\frac{j-1}{N}\sigma p_1(j)+\gamma_2 p_2(j)=0,\\
&\quad j=0,\dots,N,\quad p_1(-1)=p_1(0)=p_1(N+1)=0,\\
&\frac{\partial p_1(j,y)}{\partial y}=-\left[\lambda\frac{N-j}{N}+\frac{j-1}{N}\sigma+\mu(y)+\gamma_1\right]p_1(j,y),\qquad j=1,\dots,N,\\
&-\left[\lambda\frac{N-j}{N}+\gamma_2\right]p_2(j)+\lambda\frac{N-j+1}{N}p_2(j-1)+\gamma_0 p_0(j)+\gamma_1 p_1(j)=0,\\
&\quad j=0,\dots,N,\qquad p_2(-1)=p_1(0)=0,
\end{aligned}
\tag{1}
$$

with boundary condition

$$
p_1(j,0)=\lambda\frac{N-j+1}{N}p_0(j-1)+\frac{j}{N}\sigma p_0(j),\qquad j=1,\dots,N,
\tag{2}
$$

where $\mu(y)=\dfrac{B'(y)}{1-B(y)}$ is the service completion rate function.

Denoting the partial characteristic functions by

$$
H_k(u)=\sum_{j=0}^{N}e^{iuj}p_k(j),\quad k=0,2;\qquad H_1(u,y)=\sum_{j=1}^{N}e^{iuj}p_1(j,y),
$$

system (1) and condition (2) can be rewritten in the form

$$
\begin{aligned}
&-(\lambda+\gamma_0)H_0(u)+\left[\lambda e^{iu}-\frac{\sigma}{N}\right]H_1(u)+e^{-iu}\int\limits_0^\infty H_1(u,y)\mu(y)dy++\gamma_2 H_2(u)\\
&\quad+i\frac{(\sigma-\lambda)}{N}\frac{dH_0(u)}{du}+i\frac{(\lambda e^{iu}-\sigma)}{N}\frac{dH_1(u)}{du}=0,\\
&\frac{\partial H_1(u,y)}{\partial y}=\left[\frac{\sigma}{N}-\lambda-\mu(y)-\gamma_1\right]H_1(u,y)-i\frac{(\lambda-\sigma)}{N}\frac{\partial H_1(u,y)}{\partial u},\\
&\gamma_0 H_0(u)+\gamma_1 H_1(u)+\left[\lambda(e^{iu}-1)-\gamma_2\right]H_2(u)+i\frac{\lambda(e^{iu}-1)}{N}\frac{dH_2(u)}{du}=0,\\
&H_1(u,0)=\lambda e^{iu}H_0(u)+i\frac{(\lambda e^{iu}-\sigma)}{N}\frac{dH_0(u)}{du}.
\end{aligned}
\tag{3}
$$

To get an exact solution to this system is very complicated, that is why we solve it by the help of an asymptotic method, see [36]. The present model is a generalization of the ones treated in our previous papers either with exponentially distributed service times or with reliable server, see [29], [31], [34], [35].

4. Asymptotic analysis of the first order

For the first order solution to (3), we can state the following theorem, see [29].

Theorem 1. *Let $Q(t)$ be the number of requests in the system in steady-state, then*

$$\lim_{N\to\infty} \mathsf{E}\exp\left\{iw\frac{Q(t)}{N}\right\} = \exp\left\{iw\kappa\right\}, \tag{4}$$

where the value of parameter κ is the positive solution of the equation

$$\lambda\left(1-\kappa\right) - a(\kappa)\left[R_0(\kappa) - R_1(\kappa)\right] + \gamma_1 R_1(\kappa) = 0, \tag{5}$$

here $a\left(\kappa\right)$ is

$$a\left(\kappa\right) = \lambda\left(1-\kappa\right) + \sigma\kappa, \tag{6}$$

and the steady-state probabilities $R_k(\kappa)$ of the state of the server depend on κ. They can be determined in the following way:

$$\begin{aligned} R_0(\kappa) &= \left\{\frac{\gamma_0+\gamma_2}{\gamma_2} + \frac{\gamma_1+\gamma_2}{\gamma_2}\cdot\frac{a\left(\kappa\right)}{a\left(\kappa\right)+\gamma_1}\left[1-B^*(a(\kappa)+\gamma_1)\right]\right\}^{-1}, \\ R_1(\kappa) &= R_0(\kappa)\frac{a\left(\kappa\right)}{a\left(\kappa\right)+\gamma_1}\cdot\left[1-B^*(a(\kappa)+\gamma_1)\right], \\ R_2(\kappa) &= \frac{1}{\gamma_2}\left\{\gamma_0 R_0(\kappa) + \gamma_1 R_1(\kappa)\right\}. \end{aligned} \tag{7}$$

The proof of this theorem is given in [29]. Here we wanted to show the dependence on κ to get its value. But as soon as κ is given for the simpler notation, we omit it. Now, we present some results from the proof of Theorem 1, which we will need later on in the proof of Theorem 2.

Namely, the system of equations

$$\int_0^\infty R_1(y)\mu(y)dy = a\left(R_0 - R_1\right) + \gamma_0 R_0 - \gamma_2 R_2,$$

$$R_1^{'}(y) = -\left[a+\mu(y)+\gamma_1\right]R_1(y), \qquad \gamma_0 R_0 + \gamma_1 R_1 - \gamma_2 R_2 = 0, \qquad R_1(0) = aR_0, \tag{8}$$

and equality

$$\int_0^\infty R_1(y)\mu(y)dy = \lambda(1-\kappa) = aR_0B^*(a+\gamma_1), \tag{9}$$

where $R_1(y)$ denotes the density function of steady-state probability that the elapsed service time is less than y. The probabilistic meaning of the last equation is that the mean arrival rate equals the mean departure/service rate. Furthermore, $\int_0^\infty R_1(y)dy = R_1$.

For easier understanding, let us give some explanations to the existence of κ. If we denote by $r(\kappa)$ the left-hand side of (5), then it is easy to see that $r(0) = \lambda(1-R_0(0)) + (\lambda+\gamma_1)R_1(0) > 0$, and $r(1) = -\sigma R_0(1)B^*(\sigma+\gamma_1) < 0$. Hence, independently of the parameters N, λ, σ, γ_0, γ_1, γ_2, there exists a κ in the interval $(0,1)$.

5. Distribution of the number of transitions/attempts of a request into the orbit

It is not difficult to see that in this system a service may not be completed because of either conflicts or breakdowns of the server. In the case of interrupted service, the customer goes into the orbit and after some random delay it retries for service. That is why for these systems it is a very important and interesting task to find the distribution of the number of transitions of a customer into the orbit.

To the best knowledge of the authors, there is no paper dealing with the distribution of the number of retrials with collisions. We must mention [9] and [10], in which the distribution of the number of retrials in an $M/G/1//N$ retrial queue with reliable and unreliable server, respectively, was investigated but without collision. The author used the discrete-transformation method combined with a solution of linear systems of equations to get the exact distribution.

Let $\nu(t)$ denote the residual number of attempts of the tagged request into the orbit, that is the number of retrials from t till the completion of its successful service.

Furthermore, let us define the state of the server by

$$S(t) = \begin{cases} 0 & \text{if the server is idle at time } t, \\ 1 & \text{if the server is busy but not by tagged customer at time } t, \\ 2 & \text{if the server is broken at time } t, \\ 3 & \text{if the server is busy by the tagged customer at time } t. \end{cases}$$

Finally, let us introduce the conditional generating functions as

$$G_k(j,z) = \mathsf{E}\left\{z^{\nu(t)}|S(t)=k, Q(t)=j\right\}, \qquad k=0,2;$$
$$G_k(j,z,y) = \mathsf{E}\left\{z^{\nu(t)}|S(t)=k, Q(t)=j, y(t)=y\right\}, \qquad k=1,3. \tag{10}$$

Hence, using standard method in steady-state for functions $G_0(j,z)$, $G_1(j,z,y)$, $G_2(j,z)$, $G_3(j,z,y)$, we have the following system of Kolmogorov equations:

$$-\left[\lambda\frac{N-j}{N}+\frac{j}{N}\sigma+\gamma_0\right]G_0(j,z)+\lambda\frac{N-j}{N}G_1(j+1,z,0)+\gamma_0 G_2(j,z)$$
$$+\frac{j-1}{N}\sigma G_1(j,z,0)+\frac{\sigma}{N}G_3(j,z,0)=0,$$

$$-\left[\lambda\frac{N-j}{N}+\frac{j-1}{N}\sigma+\mu(y)+\gamma_1\right]G_1(j,z,y)+\lambda\frac{N-j}{N}G_0(j+1,z)+\gamma_1 G_2(j,z)$$
$$+\frac{j-2}{N}\sigma G_0(j,z)+\frac{\sigma}{N}zG_0(j,z)+\mu(y)G_0(j-1,z)+\frac{\partial G_1(j,z,y)}{\partial y}=0,$$

$$-\left[\lambda\frac{N-j}{N}+\frac{\sigma}{N}+\gamma_2\right]G_2(j,z)+\lambda\frac{N-j}{N}G_2(j+1,z)+\frac{\sigma}{N}zG_2(j,z)+\gamma_2 G_0(j,z)=0,$$

$$-\left[\lambda\frac{N-j}{N}+\frac{j-1}{N}\sigma+\mu(y)+\gamma_1\right]G_3(j,z,y)+\lambda\frac{N-j}{N}zG_0(j+1,z)$$
$$+\frac{j-1}{N}\sigma zG_0(j,z)+\gamma_1 zG_2(j,z)+\mu(y)+\frac{\partial G_3(j,z,y)}{\partial y}=0. \tag{11}$$

Theorem 2. *For the generating function of the number of transitions ν of the tagged customer into the orbit, we have*

$$\lim_{N\to\infty}\mathsf{E}\,z^{\nu}=\frac{1-q}{1-qz}, \tag{12}$$

where the probability q can be obtained as

$$q=1-R_0B^*(a+\gamma_1), \tag{13}$$

here

$$a=\lambda(1-\kappa)+\sigma\kappa.$$

Proof. After introducing the notation $\frac{1}{N}=\varepsilon$, performing the following substitutions in system (11), that is

$$
\begin{aligned}
j\varepsilon = x, \qquad G_k(j,z) &= F_k(x,z,\varepsilon), \qquad k = 0,2,\\
G_k(j,z,y) &= F_k(x,z,y,\varepsilon), \qquad k = 1,3,
\end{aligned} \tag{14}
$$

denoting $a(x) = \lambda(1-x) + \sigma x$, and assuming the existence of the partial derivatives, we obtain the above system in the form

$$
\begin{aligned}
&-[a(x)+\gamma_0]\,F_0(x,z,\varepsilon) + \lambda(1-x)F_1(x+\varepsilon,z,0,\varepsilon) + \gamma_0 F_2(x,z,\varepsilon)\\
&\quad + \sigma(x-\varepsilon)F_1(x,z,0,\varepsilon) + \sigma\varepsilon F_3(x,z,0,\varepsilon) = 0,\\
&-[a(x)-\sigma\varepsilon+\mu(y)+\gamma_1]\,F_1(x,z,y,\varepsilon) + \lambda(1-x)F_0(x+\varepsilon,z,\varepsilon) + \gamma_1 F_2(x,z,\varepsilon)\\
&\quad + \sigma(x-2\varepsilon)F_0(x,z,\varepsilon) + \sigma\varepsilon z F_0(x,z,\varepsilon) + \mu(y)F_0(x-\varepsilon,z,\varepsilon) + \frac{\partial F_1(x,z,y,\varepsilon)}{\partial y} = 0,\\
&-[\lambda(1-x)+\sigma\varepsilon+\gamma_2]\,F_2(x,z,\varepsilon) + \lambda(1-x)F_2(x+\varepsilon,z,\varepsilon) + \sigma\varepsilon z F_2(x,z,\varepsilon)\\
&\quad + \gamma_2 F_0(x,z,\varepsilon) = 0,\\
&-[a(x)-\sigma\varepsilon+\mu(y)+\gamma_1]\,F_3(x,z,y,\varepsilon) + \lambda(1-x)zF_0(x+\varepsilon,z,\varepsilon)\\
&\quad + \sigma(x-\varepsilon)zF_0(x,z,\varepsilon) + \gamma_1 z F_2(x,z,\varepsilon) + \mu(y) + \frac{\partial F_3(x,z,y,\varepsilon)}{\partial y} = 0.
\end{aligned} \tag{15}
$$

Stage 1. Assuming the limits under the condition $\varepsilon \to 0$, and using notations $\lim_{\varepsilon\to 0} F_k(x,z,\varepsilon) = F_k(x,z)$, $k = 0,2$, and $\lim_{\varepsilon\to 0} F_k(x,z,y,\varepsilon) = F_k(x,z,y)$, $k = 1,3$, system (15) can be rewritten as

$$
\begin{aligned}
&-[a(x)+\gamma_0]\,F_0(x,z) + a(x)F_1(x,z,0) + \gamma_0 F_2(x,z) = 0,\\
&\frac{\partial F_1(x,z,y)}{\partial y} - [a(x)+\mu(y)+\gamma_1]\,F_1(x,z,y) + [a(x)+\mu(y)]\,F_0(x,z) + \gamma_1 F_2(x,z) = 0,\\
&-\gamma_2 F_2(x,z) + \gamma_2 F_0(x,z) = 0,\\
&\frac{\partial F_3(x,z,y)}{\partial y} - [a(x)+\mu(y)+\gamma_1]\,F_3(x,z,y) + a(x)zF_0(x,z) + \mu(y) + \gamma_1 z F_2(x,z) = 0.
\end{aligned} \tag{16}
$$

It is easy to see that from the third and first equations of system (16) it follows that $F_0(x,z) = F_2(x,z)$, $F_0(x,z) = F_1(x,z,0)$, respectively, thus $F_0(x,z) = F_2(x,z) = F_1(x,z,0)$. Designating their common value by $F(x,z)$, let us consider the second equation of system (16). The solution to this equation can be written in the form

$$F_1(x,z,y) = e^{\int\limits_0^y [a(x)+\gamma_1+\mu(u)]du} \left\{ F_1(x,z,0) \right.$$
$$\left. -F(x,z) \int\limits_0^y e^{-\int\limits_0^v [a(x)+\gamma_1+\mu(u)]du} [a(x)+\gamma_1+\mu(v)]\, dv \right\}, \qquad (17)$$

from which it is not difficult to obtain that functions $F_1(x,z,y)$ and $F(x,z)$ are equal.

Then let us consider the fourth equation of system (16). Taking into account that $F(x,z) = F_0(x,z) = F_2(x,z)$, the solution to this equation can be obtained in the form

$$F_3(x,z,y) = e^{\int\limits_0^y [a(x)+\gamma_1+\mu(u)]du} \left\{ F_3(x,z,0) \right.$$
$$\left. - \int\limits_0^y e^{-\int\limits_0^v [a(x)+\gamma_1+\mu(u)]du} [(a(x)+\gamma_1) z F(x,z) + \mu(v)]\, dv \right\}. \qquad (18)$$

Executing limiting transition as $y \to \infty$, the first factor of equation (18) is equal to infinity, then consequently, the second factor equals zero. Hence,

$$F_3(x,z,0) = [a(x)+\gamma_1]\, zF(x,z) \int\limits_0^\infty e^{-\int\limits_0^v [a(x)+\gamma_1+\mu(u)]du} dv$$
$$+ \int\limits_0^\infty e^{-\int\limits_0^v [a(x)+\gamma_1+\mu(u)]du} \mu(v)dv. \qquad (19)$$

Now, after standard calculations, it is not difficult to show that

$$F_3(x,z,0) = [1 - B^*(a(x)+\gamma_1)]\, zF(x,z) + B^*(a(x)+\gamma_1). \qquad (20)$$

Stage 2. Our method is constructive, we represent the solution to system (15) in the form of a decomposition, namely

$$F_k(x,z,\varepsilon) = F(x,z) + \varepsilon f_k(x,z) + O(\varepsilon^2), \qquad k = 0,2,$$
$$F_1(x,z,y,\varepsilon) = F(x,z) + \varepsilon f_1(x,z,y) + O(\varepsilon^2),$$
$$F_3(x,z,y,\varepsilon) = F_3(x,z,y) + \varepsilon f_3(x,z,y) + O(\varepsilon^2). \qquad (21)$$

Substituting them into system (15) equating coefficients at identical degrees ε, the terms that do not contain ε are mutually canceled resulting in the following system of equations:

$$
\begin{aligned}
&-[a(x)+\gamma_0]\,f_0(x,z)+a(x)f_1(x,z,0)+\gamma_0 f_2(x,z)\\
&\quad=\sigma F(x,z)-\sigma F_3(x,z,0)-\lambda(1-x)\frac{\partial F(x,z)}{\partial x},\\
&-[a(x)+\mu(y)+\gamma_1]\,f_1(x,z,y)+[a(x)+\mu(y)]\,f_0(x,z)+\gamma_1 f_2(x,z)+\frac{\partial f_1(x,z,y)}{\partial y}\\
&\quad=\sigma(1-z)F(x,z)-[\lambda(1-x)-\mu(y)]\,\frac{\partial F(x,z)}{\partial x},\\
&\gamma_2\,[f_0(x,z)-f_2(x,z)]=\sigma(1-z)F(x,z)-\lambda(1-x)\frac{\partial F(x,z)}{\partial x},\\
&-[a(x)+\mu(y)+\gamma_1]\,f_3(x,z,y)+a(x)zf_0(x,z)+\gamma_1 zf_2(x,z)+\frac{\partial f_3(x,z,y)}{\partial y}\\
&\quad=\sigma zF(x,z)-\sigma F_3(x,z,y)-\lambda(1-x)z\frac{\partial F(x,z)}{\partial x}.
\end{aligned}
\tag{22}
$$

Let us multiply the first equation of system (22) by R_0, and the third one by R_2. Let us multiply the second equation by $R_1(y)$, and integrate from 0 to ∞. After adding the received equalities, we get

$$
\begin{aligned}
&\left\{-[a(x)+\gamma_0]\,R_0+a(x)R_1+\gamma_2R_2+\int\limits_0^\infty\mu(y)R_1(y)dy\right\}f_0(x,z)\\
&\quad+a(x)R_0f_1(x,z,0)-[a(x)+\gamma_1]\int\limits_0^\infty f_1(x,z,y)R_1(y)dy-\int\limits_0^\infty f_1(x,z,y)\mu(y)R_1(y)dy\\
&\quad+\int\limits_0^\infty R_1(y)\frac{\partial f_1(x,z,y)}{\partial y}dy+[\gamma_0R_0+\gamma_1R_1-\gamma_2R_2]\,f_2(x,z)\\
&=\sigma(1-z)F(x,z)+\sigma zR_0F(x,z)-\sigma R_0F_3(x,z,0)\\
&\quad+\left\{\int\limits_0^\infty\mu(y)R_1(y)dy-\lambda(1-x)\right\}\frac{\partial F(x,z)}{\partial x}.
\end{aligned}
\tag{23}
$$

Then, it can be shown that

$$
\int\limits_0^\infty R_1(y)\frac{\partial f_1(x,z,y)}{\partial y}dy=-R_1(0)f_1(x,z,0)-\int\limits_0^\infty f_1(x,z,y)R_1'(y)dy. \tag{24}
$$

Due to the replacement $\dfrac{j}{N} = x$, we can conclude that $x = \kappa$. Keeping this fact in mind, the first term of equality (23) is equal to zero, due to the first equation of system (8).

Similarly, taking into account that $\int\limits_0^\infty \mu(y)R_1(y)dy = \lambda(1-\kappa)$ from (9), and $\gamma_0 R_0 + \gamma_1 R_1 - \gamma_2 R_2 = 0$ from system (8), we can rewrite (23) in the form

$$\{a(\kappa)R_0 - R_1(0)\}\, f_1(z,0) - \int\limits_0^\infty f_1(z,y)\Big\{(a(\kappa)+\gamma_1)\, R_1(y) + \mu(y)R_1(y) + R_1^{'}(y)\Big\}\, dy$$
$$= \sigma(1-z)F(z) + \sigma R_0\left[zF(z) - F_3(z,0)\right]. \tag{25}$$

From the second equation of system (8), it follows that the integral is equal to zero. From the fourth equation of system (8), we see that the factor of function $f_1(z,0)$ is also equal to zero. Thus, we obtain

$$(1-z)F(z) + R_0\left[zF(z) - F_3(z,0)\right] = 0. \tag{26}$$

Let substitute the explicit form of function $F_3(z,0)$, which is determined by equality (20), and keeping in mind that $x = \kappa$, we get

$$(1-z)F(z) + R_0\left\{zF(z) - \left[1 - B^*(a(\kappa)+\gamma_1)\right] zF(z) - B^*(a(\kappa)+\gamma_1)\right\} = 0, \tag{27}$$

from which it is not difficult to obtain that

$$F(z) = \frac{1-q}{1-qz}, \tag{28}$$

where $q = 1 - R_0 B^*(a(\kappa)+\gamma_1)$.

Hence, by the law of total probability for the probability generating function of the number of transitions of a customer into the orbit, we have

$$Ez^\nu = R_0 F_3(z,0) + (1-R_0)zF(z). \tag{29}$$

From equation (26), it follows that $R_0F_3(z,0) = (1-z)F(z) + R_0 zF(z)$. Thus, equation (29) can be rewritten as

$$Ez^\nu = (1-z)F(z) + R_0 zF(z) + (1-R_0)zF(z) = F(z). \tag{30}$$

Hence, we obtain that

$$Ez^\nu = \frac{1-q}{1-qz}, \tag{31}$$

coinciding with our statement given in (12). □

Consequently, ν is geometric, namely

$$P\{\nu = n\} = (1-q)q^n, \qquad n = \overline{0,\infty}, \tag{32}$$

and for the prelimit situation, that is, when N is fixed, we can and will use the following approximation $P\{\nu = n\} \approx (1-q)q^n$.

In connection with the assumptions, in the mathematical operations we would like to add the following remarks.

Remarks. The system is very complicated due to the collisions and server's breakdowns, and the authors cannot prove the existence of the above-mentioned partial derivatives and limits. The form of the Taylor expansion is based on the previous research experience of the authors. The correctness of the assumption is justified by comparing the simulation and asymptotic results presented in Section 7, that is, by an informal way of the proof.

6. Waiting time of the tagged customer in the orbit

Laplace transform of the waiting time of a tagged customer was investigated in the case of a reliable server under exponentially distributed service time in [12] without collision. For the moments of the waiting time, recursive relations were derived, but the type of the distribution could be obtained. In [47] and [50], the Laplace transform and moment of the waiting time were investigated in the case of an unreliable server but without collision.

Let W denote the waiting time of the tagged customer in the orbit. Owing to Theorem 2 in prelimit situation, that is, when N is fixed, we can state the following Theorem.

Theorem 3. *The characteristic function of the waiting time W of the tagged customer in the orbit can be obtained as*

$$\mathsf{E}e^{iuW} \approx (1-q) + q\frac{\sigma(1-q)}{\sigma(1-q) - iuN}. \tag{33}$$

Proof. Since the characteristic function of the inter-retrial time τ of the tagged customer in the orbit is

$$\mathsf{E}e^{iu\tau} = \frac{\sigma}{\sigma - iuN}, \tag{34}$$

by using the law of total expectation, we get

$$\begin{aligned}\mathsf{E}e^{iuW} &\approx \sum_{n=0}^{\infty}\left(\frac{\sigma}{\sigma - iuN}\right)^n (1-q)q^n = \frac{1-q}{1-q\frac{\sigma}{\sigma-iuN}} \\ &= (1-q)\frac{\sigma - iuN}{\sigma(1-q)-iuN} = (1-q) + q\frac{\sigma(1-q)}{\sigma(1-q)-iuN}. \end{aligned} \tag{35}$$

Hence,

$$\mathsf{E}(W) \approx \frac{N}{\sigma}\frac{q}{1-q} = \frac{N}{\sigma}\frac{1-R_0B^*(a+\gamma_1)}{R_0B^*(a+\gamma_1)}. \tag{36}$$

The theorem is proved. □

Consequently,

$$\lim_{N\to\infty} \mathsf{E}e^{iu\frac{W}{N}} = (1-q) + q\frac{\sigma(1-q)}{\sigma(1-q)-iu}.$$

Thus, by using equations (6) and (9), we get

$$\lim_{N\to\infty} \mathsf{E}(\frac{W}{N}) = \frac{1}{\sigma}\frac{q}{1-q} = \frac{1}{\sigma}\frac{1-R_0B^*(a+\gamma_1)}{R_0B^*(a+\gamma_1)} = \frac{\kappa}{\lambda(1-\kappa)}. \tag{37}$$

This type of distribution is called a *generalized exponential distribution*, as we met in our previous papers [32]–[34] and [41].

7. Average sojourn time of the customer under service

Another important characteristic of retrial queueing systems is the sojourn time, or the total service time, of a customer under service. For classical retrial queueing systems without interruptions, this characteristic can be found easily. But for the considered system in which conflicts of the customers and server failures are possible, the sojourn time of a customer has a rather complex structure. It is the sum of the following terms: a term of zero duration if a request from the orbit finds the server busy, a non-zero term of the services interrupted by conflicts and failures of the server, and finally a single term of successful service completion after that the tagged request departs from the service.

Let $\bar{T}_S$ denote the average sojourn time of a customer under service, and $T_S(t)$ the residual sojourn time of a customer under service, respectively. Let us introduce the supplementary random variable $z(t)$ equal to the residual service time, that is, the time interval from moment t until the successful completion of the service. When $N \to \infty$, we have the following statement:

Theorem 4. *The average sojourn time $\bar{T}_S$ of the customer under service can be approximated by*

$$\bar{T}_S \approx \frac{1 - B^*(a+\gamma_1)}{(a+\gamma_1)B^*(a+\gamma_1)}, \tag{38}$$

where

$$a = \lambda\,(1-\kappa) + \sigma\kappa.$$

Proof. Let us define the following function of conditional average residual sojourn time of a customer under service as follows:

$$g(z) = E\left\{T_S(t)|z(t) = z\right\}.$$

Applying the law of total probability, we can write

$$g(z) = \left[1 - (a+\gamma_1)\Delta t\right](\Delta t + g(z-\Delta t)) + (a+\gamma_1)\Delta t\bar{T}_S + o(\Delta t). \tag{39}$$

Let us execute the limiting transition under conditions $\Delta t \to 0$, then we can rewrite equation (39) in the form

$$g'(z) = -(a+\gamma_1)g(z) + 1 + (a+\gamma_1)\bar{T}_S.$$

Thus, we obtain a Cauchy problem with initial condition $g(0) = 0$. Its solution can be written in the form

$$g(z) = e^{-(a+\gamma_1)z}\int\limits_0^z e^{(a+\gamma_1)x}\left[1+(a+\gamma_1)\bar{T}_S\right]dx = \frac{1+(a+\gamma_1)\bar{T}_S}{(a+\gamma_1)}\left(1-e^{-(a+\gamma_1)z}\right). \tag{40}$$

Thus, by using the law of total probability, we obtain

$$\bar{T}_S = (1-R_0)\,\bar{T}_S + R_0\int\limits_0^\infty g(z)dB(z).$$

Substituting the explicit form (40) of function $g(z)$ and carrying simple calculations, we get

$$\bar{T}_S = \mathsf{E}(T_S) \approx \frac{1 - B^*(a+\gamma_1)}{(a+\gamma_1)B^*(a+\gamma_1)},$$

coinciding with (38).

The theorem is proved. □

Let us check our result by using Little's formula. If we apply it to $\mathsf{E}(T_S)$, we have

$$\lambda(1-\kappa)\frac{1-B^*(a+\gamma_1)}{(a+\gamma_1)B^*(a+\gamma_1)} = aR_0B^*(a+\gamma_1)\frac{1-B^*(a+\gamma_1)}{(a+\gamma_1)B^*(a+\gamma_1)} = R_1.$$

If T denotes the sojourn time of a customer in the system, then $T = W + T_S$, and using again Little's formula for the response time, we have

$$\lim_{N\to\infty} \lambda(1-\kappa)\mathsf{E}\left(\frac{W+T_S}{N}\right) = \kappa.$$

Since

$$\lim_{N\to\infty} \lambda(1-\kappa)\mathsf{E}\left(\frac{T_S}{N}\right) = 0,$$

thus

$$\lim_{N\to\infty} \mathsf{E}\left(\frac{W}{N}\right) = \frac{\kappa}{\lambda(1-\kappa)},$$

which we got in (37).

8. Numerical examples for comparisons

Let us examine the range of applicability and accuracy of the obtained asymptotic results. In [45], the simulation of the considered system was carried out, analysis of the performance measures, graphs and tables were presented. In the present paper, we use the results of the simulation in order to find out how close the asymptotic results are to the simulated ones.

Let us denote by $P_{as}(n)$ the asymptotic geometric distribution (32), and by $P_s(n)$ the probability distribution of the number of transitions of the tagged customer into the orbit obtained with the help of the simulation program. Let us define the accuracy (error) of the approximation by means of the Kolmogorov distance Δ in the following form

$$\Delta = \max_{0\le i\le N}\left|\sum_{n=0}^{i}[P_s(n)-P_{as}(n)]\right|,$$

suggesting an acceptable error $\Delta < 0.05$.

For numerical comparisons, we choose a gamma-distributed service time S with shape parameter α and scale parameter β, with Laplace–Stieltjes transform $B^*(\delta)$ of the form

$$B^*(\delta) = \left(1+\frac{\delta}{\beta}\right)^{-\alpha}.$$

It can be shown that

$$\mathsf{E}(S) = \frac{\alpha}{\beta}, \qquad \mathrm{Var}(S) = \frac{\alpha}{\beta^2}, \qquad V_S^2 = \frac{1}{\alpha},$$

where V_S^2 denotes the squared coefficient of variation of S. This distribution allows us to show the effect of the distribution on the main performance measures, because in the case when $\alpha = \beta$, that is, when the average service time is equal to the unit dealing with the same mean, we can see the impact of the variance, too.

Running the simulation program with inputs

$$\lambda = 1, \qquad \sigma = 1, \qquad \gamma_0 = 0.1, \qquad \gamma_1 = 0.1, \qquad \gamma_2 = 1,$$

and applying the proposed approximation (32), we calculate the Kolmogorov distance Δ for various values of N and $\alpha = \beta$ in Table 1.

	$N = 10$	$N = 30$	$N = 50$	$N = 70$	$N = 100$
$\alpha = 0.5$	0.0218	0.0067	0.0038	0.0029	0.0021
$\alpha = 1$	0.0292	0.0099	0.0064	0.0048	0.0035
$\alpha = 2$	0.0360	0.0119	0.0075	0.0056	0.0040

Table 1. Kolmogorov distance between distributions $P_s(n)$ and $P_{as}(n)$ for various values of parameters N and $\alpha = \beta$.

From the presented Table 1, it is clear that already at $N = 10$ for different values of α we obtain an error less than 0.04. With increasing N the error reduces, as it was expected. Thus, we can conclude that the proposed approximation very well approximates the distribution obtained using the simulation program, and it is applicable even for small values of N.

In the next Table, we can compare the mean number of retrials in prelimit case obtained by simulation and the limit distribution determined before.

	$N = 30$	$N = 50$	$N = 70$	$N = 100$	limiting
$\alpha = 0.5$	1.717	1.733	1.739	1.745	1.757
$\alpha = 1$	2.340	2.368	2.380	2.389	2.410
$\alpha = 2$	2.939	2.981	3.000	3.014	3.045

Table 2. Mean number of retrials in prelimit and limiting situations for various values of parameters N and $\alpha = \beta$.

Again, we can see the effectiveness of the proposed approximation, since the difference is very small, and of course, as N increases, the mean number of retrials in prelimit case also increases and is approaching to the limiting case, as it was expected.

Next, let us consider how the input parameters of the system influence the system characteristics such as the mean sojourn time of the customer under service $\bar{T}_S = \mathsf{E}(T_S)$, and the limiting normalized mean waiting time of the customer in the orbit $\bar{W}_o = \lim_{N\to\infty} \mathsf{E}\left(\frac{W}{N}\right)$.

For Table 3, we have the following input parameters:

$$\lambda = 2, \qquad \sigma = 10, \qquad \gamma_2 = 1, \qquad \alpha = \beta, \qquad \gamma_0 = \gamma_1.$$

	$\gamma_0 = \gamma_1 = 0.1$		$\gamma_0 = \gamma_1 = 1$		$\gamma_0 = \gamma_1 = 10$	
	$\bar{T}_S$	$\bar{W}_o$	$\bar{T}_S$	$\bar{W}_o$	$\bar{T}_S$	$\bar{W}_o$
$\alpha = 0.1$	0.124	0.103	0.086	0.285	0.037	2.100
$\alpha = 0.5$	0.422	0.608	0.376	1.354	0.272	9.756
$\alpha = 1$	1	1.846	1	3.940	1	33.74
$\alpha = 2$	3.394	7.122	3.686	15.20	5.995	198.5
$\alpha = 5$	24.34	53.44	30.20	126.5	156.1	5154

Table 3. $\bar{T}_S$ and $\bar{W}_o$ for various values of failure parameters $\gamma_0 = \gamma_1$.

From Table 3, the following conclusions could be drawn:

- With an increase of the service parameters $\alpha = \beta$, the values of $\bar{T}_S$ and $\bar{W}_o$ increase.
- The values of $\bar{W}_o$ increase as failure intensity $\gamma_0 = \gamma_1$ increases.
- The value of $\bar{T}_S$ with an increase of failure intensity of the server for $\alpha < 1$ decreases, and for $\alpha > 1$ increases.

The first two observations are natural, but the third one is surprising. Our explanation for this interesting behavior is the following: since the service time is gamma-distributed for $\alpha < 1$, there is a high probability of occurrence of small values of the service time, the smaller the α, the greater the probability of short successful service time is, and this fact influences the mean total service time. Both interrupted and successful service times can be very small with high probability, that is why $\bar{T}_S$ is less than the mean required service which is the unit. As the failure intensity increases, more and more retrials happen, that is, more and more service takes place and their duration will be shorter and shorter. When $\alpha = 1$, we have an exponential distribution with a unit mean service time. Finally, in the case of $\alpha > 1$, the service times are longer and longer, resulting in more and more phases for the interrupted service time as failure intensities increase.

It should be mentioned that this phenomenon was noticed in [30] when the mean total service time was investigated by the help of stochastic simulation.

9. Conclusion

In this paper, the waiting time of an $M/G/1//N$ retrial queueing system with conflict of requests where the server is unreliable was investigated. An asymptotic analysis was used when N tends to infinity. It was shown that the limiting probability distribution of the number of transitions of the requests into the orbit was geometric. In limiting and prelimit situation the waiting time of a customer in the orbit was generalized exponentially distributed. Parameters of these distributions were found. Moreover, the average total service time of a customer was obtained. To illustrate the power of the proposed asymptotic approach for comparisons of the results, a simulation program was developed, by the help of which the distribution of the number of transitions of a customer into the orbit was obtained and the mean and variance of the waiting time were estimated. Several sample examples were discussed in detail which demonstrated the effectiveness of the asymptotic results at finite values of N, that is, in prelimit situation.

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