Asymptotic Reliability Analysis of Complex Telecommunication Systems with Bursty Arrivals and Service*

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1 Introduction

This paper deals with a First-Come, First-Served (FCFS) queueing model to analyze the asymptotic behaviour of a heterogeneous finite-source communication system with a single processor. The sources and the processor are supposed to operate in independent random environments, respectively. Each message is characterized by its own exponentially distributed source and processing time with parameter depending on the state of the corresponding environment. Assuming that the arrival rates of the messages are many times greater than their service rates (”fast” arrival), it is shown that the time to the first system failure converges in distribution, under appropriate norming, to an exponentially distributed random variable. Some simple examples are considered to illustrate the effectiveness of the method proposed comparing the approximate characteristics to the exact ones. This paper generalizes the results of Sztrik and Rigo (Sztrik and Rigó 1993) where the sources are homogeneous and the whole system is governed by two random environments. The technique used here is similar to the one applied in Kouvatsos (Sztrik and Kouvatsos 1991) and Sztrik (Sztrik and Rigó 1993) and it is based on the theoretic investigations of Anisimov (Anisimov 1970; Anisimov 2000).

2 The Queueing Model

Consider a finite-source communication system with N heterogeneous sources and a single processor. The sources and the receiver operate in independent random environments. The environmental changes are reflected in the values of the acces and service rates that prevail at any point of time. The main objective is to adapt these parameters to respond to random changes effectively and thus maintain derived level of system performance.

Source p is assumed to operate in a random environment governed by an ergodic Markov chain $(\xi_p(t), t \geq 0)$ with state space $(1, \ldots, r_p)$ and with transition density matrix

$$\left(a^{(p)}_{i_p j_p}, i_p, j_p = 1, \ldots, r_p, a^{(p)}_{i_p i_p} = \sum_{k \neq i_p} a^{(p)}_{i_p k} \right).$$

Whenever the environmental process $\xi_p(t)$ is in state i_p the probability that the source generates a request in the time interval $(t, t+h)$ is $\lambda_p(i_p, \varepsilon) + o(h)$, $p = 1, \ldots, N$. Each message is transmitted to a receiver where the service immediately starts if the processor is idle, otherwise a queueing line is formed. The service discipline is First-Come, First-Served (FCFS). The receiver is also supposed to operate in a random environment governed by an ergodic Markov chain $(\xi(t), t \geq 0)$ with state space $(1, \ldots, r_{N+1})$ and with transition density matrix

$$\left(a^{(N+1)}_{i_{N+1} j_{N+1}}, i_{N+1}, j_{N+1} = 1, \ldots, r_{N+1}, a^{(N+1)}_{i_{N+1} i_{N+1}} = \sum_{k \neq i_{N+1}} a^{(N+1)}_{i_{N+1} k} \right).$$

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Whenever the environmental process $\xi_{N+1}(t)$ is in state i_p the probability that the service of message p is completed in time interval $(t, t+h)$ is $\mu_p(i_{N+1})h + o(h)$. If a given source has sent a message it stays idle and it can not generate another one. After being serviced each message immediately returns to its source which hence becomes active. All random variables involved here and the random environments are supposed to be independent of each other.

In practical applications it is very important to know the distribution of time until the receiver becomes empty, which is actually the busy period of the processor. For example, if it is a satellite, an aircraft or any flying object which needs orientation.

Let us consider the system under the assumption of "fast" arrivals, i.e., $\lambda_p(i_p, \varepsilon) \to \infty$ as $\varepsilon \to 0$. For simplicity let $\lambda_p(i_p, \varepsilon) = \lambda_p(i_p)/\varepsilon$, $p = 1, 1, \ldots, N$.

Denote by $Y_\varepsilon(t)$ the number of active sources at time t, and let

$$\Omega_\varepsilon(m) = inf\{t : \quad t > 0, Y_\varepsilon(t) = m + 1/Y_\varepsilon(0) \leq m\},$$

that is, the instant at which the number of active sources reaches the $(m+1)$-th level for the first time, provided that at the beginning their number is not greater than m, $m = 0, \ldots, N-1$. In the following $\Omega_\varepsilon(m)$ is referred to as the time to the first system failure. In particular, for $m = N-1$ we get the case when the processor becomes idle, i.e. $\Omega_\varepsilon(N-1)$ is the busy period length of the receiver, or processor.

Denote by $\pi_0(i_1, i_2, \ldots, i_N, i_{N+1} : 0; k_1, \ldots, k_N)$ the steady-state probability that random environment $\xi_p(t)$ is in state i_p , $p = 1, \ldots, N+1$, there is no active source and the order of arrival of messages to the processor is $(k_1, \ldots, k_N)$. Similarly, denote by $\pi_0(i_1, i_2, \ldots, i_N, i_{N+1} : 1; k_2, \ldots, k_N)$ the steady-state probability that the p-th random environment is in state i_p , $p = 1, \ldots, N+1$, source k_1 is active and the other sources sent their messages in the order $(k_2, \ldots, k_N)$. Clearly, $(k_s, \ldots, k_N) \in V_N^{N-s+1}$, $s = 1, 2$ where V_N^{N-s+1} denotes the set of all variations of order $N-s+1$ of elements $1, \ldots, N$. Now we have:

Theorem 1 *For the system in question under the above assumptions, independently of the initial state, the distribution of the normalized random variable $\varepsilon^m \Omega_\varepsilon(m)$ converges weakly to an exponentially distributed random variable with parameter*

$$\Lambda = \sum_{i_1=1}^{r_1} \cdots \sum_{i_{N+1}=1}^{r_{N+1}} \sum_{(k_1,\ldots,k_N)\in V_N^N} \pi_0(i_1, \ldots, i_{N+1} : 1; k_2, \ldots, k_N)$$
$$\times \frac{\mu_{k_2}(i_{N+1})}{\lambda_{k_1}(i_{k_1})} \times \frac{\mu_{k_3}(i_{N+1})}{\lambda_{k_1}(i_{k_1}) + \lambda_{k_2}(i_{k_2})} \times \ldots \times \frac{\mu_{k_{m+1}}(i_{N+1})}{\lambda_{k_1}(i_{k_1}) + \ldots + \lambda_{k_m}(i_{k_m})} \frac{1}{D},$$

where

$$D = \sum_{p=1}^{N+1} \sum_{j_p=1}^{r_p} \sum_{\substack{i_p \neq j_p}}^{r_p} \sum_{(k_1,\ldots,k_n)\in V_N^N} \pi_0(i_1, \ldots, i_{N+1} : 0; k_1, \ldots, k_N)$$
$$\times \frac{a^{(p)}_{i_p j_p}}{\left(\sum_{q=1}^{N+1} a^{(q)}_{i_q i_q} + \mu_{k_1}(i_{N+1})\right)^2}.$$

Proof. The idea of the proof is the following. It is shown the the describing multi-component Markov chain has a monotone structure and its state space forms an S-set. Then the main part of the stationary probabilities can be obtained recursively. Since the whole procedure is similar to the one used in Sztrik (Sztrik and Kouvatsos 1991; Sztrik and Rigó 1993) and due to the limited page numbers only the crutial parts are discussed.

Let us introduce the following stochastic process

$$Z_\varepsilon(t) = (\xi_1(t), \ldots, \xi_{N+1}(t) : Y_\varepsilon(t); \beta_1(t), \ldots, \beta_{Y_\varepsilon(t)}),$$

where $\beta_1(t), \ldots, \beta_{Y_\varepsilon(t)}$ denote the indices of the messages in order of their arrival at the processor. It is easy to see that $(Z_\varepsilon(t), t \geq 0)$ is a multi-dimensional Markov chain with rather complex state space

$$E = ((i_1, \ldots, i_{N+1} : s; k_1, \ldots, k_{N-s}), i_p = 1, ..., r_p, p = 1, \ldots, N+1,$$
$$(k_1, \ldots, k_{N-s}) \in V_N^{N-s}, s = 0, \ldots, N),$$

where $k_0 = \{0\}$ by definition. Furthermore, let

$$\langle \alpha_m \rangle = ((i_1, \ldots, i_{N+1} : s; k_1, \ldots, k_{N-s}), i_p = 1, ..., r_p, p = 1, \ldots, N+1,$$
$$(k_1, \ldots, k_{N-s}) \in V_N^{N-s}, s = 0, \ldots, m)$$

be a subset of the states.

Let us denote by $g_\varepsilon(\langle \alpha_m \rangle)$ the steady state probability of exit from $\langle \alpha_m \rangle$, that is

$$g_\varepsilon(\langle \alpha_m \rangle) = \sum_{i^{(m)} \in X_m} \pi_\varepsilon\left(i^{(m)}\right) \sum_{j^{(m+1)} \in X_{m+1}} p_\varepsilon\left(i^{(m)}, j^{(m+1)}\right).$$

Hence our aim is to determine the distribution of the first exit time of $Z_\varepsilon(t)$ from $\langle \alpha_m \rangle$, provided that $Z_\varepsilon(t) \in \langle \alpha_m \rangle$.

First we have to give the transition probabilities for the embedded Markov chain and then the limits as $\varepsilon \to 0$. After that a systems of equations is derived for the probabilities

$$\pi_0(i_1, \ldots, i_{N+1} : 0; k_1, \ldots, k_N), \quad \pi_0(i_1, \ldots, i_{N+1} : 1; k_1, \ldots, k_{N-1}),$$

$i_p = 1, \ldots, r_p$, $p = 1, \ldots, N+1$, $(k_1, \ldots, k_{N-s}) \in V_N^{N-s}$, $s = 0, 1$.

To apply the results of Anisimov (Anisimov 1970; Anisimov 2000) we need the solution of this systems with normalizing condition

$$\sum_{p=1}^{N+1} \sum_{i_p=1}^{r_p} \sum_{(k_1, \ldots, k_n)} \{\pi_0(i_1, \ldots, i_{N+1} : 0; k_1, \ldots, k_N) + \pi_0(i_1, \ldots, i_{N+1} : 1; k_1, \ldots, k_{N-1})\} = 1.$$

Suppose that we have this solution. Then we get

$$g(\langle \alpha_m \rangle) = \varepsilon^m \sum_{i_1=1}^{r_1} \cdots \sum_{i_{N+1}=1}^{r_{N+1}} \sum_{(k_1, \ldots, k_N) \in V_N^N} \pi_0(i_1, \ldots, i_{N+1} : 1; k_2, \ldots, k_N)$$
$$\times \frac{\mu_{k_2}(i_{N+1})}{\lambda_{k_1}(i_{k_1})} \times \frac{\mu_{k_3}(i_{N+1})}{\lambda_{k_1}(i_{k_1}) + \lambda_{k_2}(i_{k_2})} \times \ldots \times \frac{\mu_{k_{m+1}}(i_{N+1})}{\lambda_{k_1}(i_{k_1}) + \ldots + \lambda_{k_m}(i_{k_m})} (1 + o(1)).$$

Taking into account the exponentiality of $\tau_\varepsilon(i_1, \ldots, i_{N+1} : s; k_1, \ldots, k_{N-s})$ for fixed Θ we have

$$\mathbb{E}exp\{i\varepsilon^m \Theta \tau_\varepsilon(i_1, \ldots, i_{N+1} : 0; k_1, \ldots, k_{N-s})\}$$
$$= 1 + \varepsilon^m \frac{i\Theta}{a_{i_1 i_1}^{(1)} + \ldots + a_{i_{N+1} i_{N+1}}^{(N+1)} + \mu_{k_1}(i_{N+1})} (1 + o(1)).$$

Notice that $\beta_\varepsilon = \varepsilon^m$ and therefore we immediately get that $\varepsilon^m \Omega_\varepsilon(m)$ converges weakly to an exponentially distributed random variable with parameter

$$\Lambda = \sum_{i_1=1}^{r_1} \cdots \sum_{i_{N+1}=1}^{r_{N+1}} \sum_{(k_1, \ldots, k_N) \in V_N^N} \pi_0(i_1, \ldots, i_{N+1} : 1; k_2, \ldots, k_N)$$
$$\times \frac{\mu_{k_2}(i_{N+1})}{\lambda_{k_1}(i_{k_1})} \times \frac{\mu_{k_3}(i_{N+1})}{\lambda_{k_1}(i_{k_1}) + \lambda_{k_2}(i_{k_2})} \times \ldots \times \frac{\mu_{k_{m+1}}(i_{N+1})}{\lambda_{k_1}(i_{k_1}) + \ldots + \lambda_{k_m}(i_{k_m})} \frac{1}{D}.$$

which completes the proof.

Consequently, when $m = N-1$, we get that the busy period length of the receiver is asymptotically an exponentially distributed random variable with parameter $\varepsilon^{N-1}\Lambda$.

In particular, for homogeneous service we have:

$$\varepsilon^{N-1}\Lambda = \varepsilon^{N-1}\frac{1}{N!}\sum_{i_1=1}^{r_1}\cdots\sum_{i_{N+1}=1}^{r_{N+1}}\sum_{(k_1,\ldots,k_N)\in V_N^N}\left(\pi_{i_1}^{(1)}\ldots\pi_{i_{N+1}}^{(N+1)}\right)\mu(i_{N+1})$$
$$\times\frac{\mu(i_{N+1})}{\lambda_{k_1}(i_{k_1})}\times\frac{\mu(i_{N+1})}{\lambda_{k_1}(i_{k_1})+\lambda_{k_2}(i_{k_2})}\times\ldots\times\frac{\mu(i_{N+1})}{\lambda_{k_1}(i_{k_1})+\ldots+\lambda_{k_{N-1}}(i_{k_{N-1}})}.$$

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