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THRESHOLD POLICIES FOR CONTROLLED RETRIAL QUEUES WITH HETEROGENEOUS SERVERS

by

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Threshold Policies for Controlled Retrial Queues with Heterogeneous Servers [1]

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Abstract

Retrial queues are important stochastic models for many telecommunication systems. In order to construct competitive networks it is necessary to investigate problems related to optimal control of queueing systems. This paper considers K -server retrial systems with Markovian arrival process, heterogeneous service time distributions of general phase-type and exponentially distributed retrial times. It is shown that the optimal policy which minimizes the mean number of customers in the system is of a threshold type with threshold levels depending on the states of the arrival, retrial and service processes. Based on the Howard's iteration algorithm a numerical procedure for an optimal control is proposed. Finally, some numerical results are given to illustrate the system's dynamics.

AMS subject classification: 60K25, 93E20
Key words: Optimal control, MAP, PH, retrial queueing system, controllable queueing systems, monotonicity of optimal policies, threshold levels, numerical analysis

1 Introduction

Retrial queueing models are effective tools to describe the operation of many telecommunication networks. Since the theory of controllable queueing systems has many applications involving the control of admission, servicing, routing and

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scheduling of jobs in queue and networks it seems to be very promising to match these two approaches. This paper deals with retrial queues with multiple heterogeneous servers where the arriving customers form a Markovian arrival process (MAPs, see [13] and [10]), service time distributions are assumed to be of general phase type (PH, see [14]) and the retrial times are supposed to be exponentially distributed random variables. As usual, the arrivals, retrials and service times are assumed to be independent random variables.

Recent investigations for retrial $MAP/PH/K$ systems with homogeneous servers are available (e.g. [4, 2, 12]), as well as statistical model fitting for MAPs and PH distributions (see [1] and [3]). Since results in the analysis of these traditional or so called non-controllable retrial queueing models, where the arriving customer is directed to the orbit if and only if all servers are busy, have already been achieved, our aim is to combine the traditional and the controlled retrial queues with heterogeneous servers and to find some optimal control policies.

Controlled queues are assumed to involve a so-called decision maker or controller. Looking at the state of the system the controller may considerably improve the system's performance by reducing the queue length or increasing the throughput, whereas in the absence of a controller the system's behaviour may get quite erratic, exhibiting periods of high load and long queues followed by periods during which the servers remain idle. Therefore, it is clear that it may be better, e.g. in terms of average number of jobs in the system, not to start a service on a slow server whenever the current number of customers in the orbit is not too large, so that the waiting customer can anticipate being serviced on the fast server within a short delay. The theoretical foundations of controlled queueing systems have been developed within the theory of Markov, semi-Markov and semi-regenerative decision processes [7, 8, 17, 22, 20, 24].

The problem of an optimal allocation of jobs between heterogeneous servers aiming to minimize the mean number of jobs in the ordinary queueing system was considered in [5, 9, 11, 16, 18, 21, 23]. It was shown that the optimal policy belongs to a class of structured policies, i.e. threshold policies, which use a slow server only when the queue length exceeds a certain threshold.

To the best knowledge of the authors no paper on controlled retrial queueing systems has been published, thus our goal in this paper is to show that a threshold policy is optimal for retrial queues as well, furthermore an algorithm is proposed which allows us to construct these optimal policies. Several numerical examples are given to illustrate the effect of different input parameters on threshold functions by the help of which the optimal control policies are obtained.

2 Problem description

Consider a retrial $MAP/PH/K$ queueing system with K heterogeneous servers. The service time (ST) distributions are supposed to be of phase type with representations (η_k, M_k). The dimension of the PH-distribution for the k-th server is denoted by m_k. The vectors $\eta_k = (\eta_k^1, \ldots, \eta_k^{m_k})$ are the initial states of the phase-type service processes and the irreducible matrices $M_k = [\mu_k^{ij}]$ contain those transition intensities which do not lead to service completion. The intensities of transitions, which lead to service completion, are defined by the vectors $\vec{\mu}_k = -M_k\vec{1}$.

The Markovian arrival process is parametrized by the rate matrices $\Lambda = [\lambda_{ij}]$ (which specifies intensities of phase transitions without arrivals) and $N = [\nu_{ij}]$ (which specifies intensities of phase transitions accompanied by an arrival), whose sum $\Lambda + N$ is an irreducible infinitesimal generator of order m_{K+1}. The average arrival rate $\overline{\lambda}$ is defined as $\overline{\lambda} = \vec{\pi}N\vec{1}$, where $\vec{\pi}$ is the invariant vector of the stationary distribution of the arrival process.

The vector $\vec{\pi}$ is a unique solution to the system $\vec{\pi}(\Lambda + N) = \vec{0}$, $\vec{\pi}\vec{1} = 1$. Here $\vec{1}$ is the column-vector of appropriate size consisting of ones and $\vec{0}$ is the row-vector of appropriate size consisting of zeros. For more information on MAPs and PH-distributions, see [10] and [14], respectively.

It is assumed that the times between the successive retrials of each jobs are exponentially distributed with parameter γ, thus the total retrial rate γ_i depend on the current number i of customers in the orbit, that is $\gamma_i = i\gamma$ (cf. [4], section 6). Denote by $B < \infty$ the maximal possible number of customers in the orbit.

The control epochs are the arrival times of new or retrial customers. At the arrival times of new customers the control consists in sending them to one of the idle servers or to the orbit if it is not full. When retrial arrivals take place the control consists in either sending a customer to some idle server or leaving all customers in the orbit. An arriving customer is rejected only in the case if at the time of its arrival the orbit is full and all servers are busy. A customer starting service on a slow server has to complete service there slow, even when a faster server may become available during its service time.

We assume there are K servers with mean service times

$$0 < \overline{\mu}_1^{-1} \leq \overline{\mu}_2^{-1} \leq \cdots \leq \overline{\mu}_K^{-1}, \tag{1}$$

where $\overline{\mu}_j^{-1} = -\eta_j^T M_j^{-1}\vec{1}$, i.e. the fastest (in average) server has the lowest index.

To model the system dynamics consider the controllable process

$$\{Z(t)\} = \{X(t), U(t)\}.$$

Here, the process

$$\{X(t)\}_{t\geq 0} = \{D_0(t), D_1(t), \dots, D_K(t), D_{K+1}(t)\}$$

denotes the observed process and it is a vector with the following components:

$D_0(t)$ is the number of customers in the orbit at time t,
$D_1(t), \dots, D_K(t)$ describe the phases of the servers at this time,

$$D_j(t) = \begin{cases} 0, & \text{if the } j\text{-th server is idle at time } t \text{ and} \\ d_j = \overline{1, m_j}, & \text{if the } j\text{-th server is in phase } d_j, \end{cases}$$

$D_{K+1}(t) = \{d_{K+1} = \overline{1, m_{K+1}}\}$ describes the phase of the arrival process.
Denote the state space of the observed process by

$$E = \mathbf{N} \times \prod_{k=1}^{K} \{0, \dots, m_k\} \times \{1, \dots, m_{K+1}\}$$

with $\mathbf{N} := \{0, 1, \dots, B\}$. For each state $x = (d_0, d_1, \dots, d_K, d_{K+1})$ denote by $q(x) := d_0(x)$, $d_j(x)$ and $d_{K+1}(x)$ the number of jobs in the orbit, the states of ST-phases for each server ($j = \overline{1, K}$) and the MAP in system state x, respectively. Also, denote by $J_0(x)$ and $J_1(x)$ the sets of indices j for which $d_j(x) = 0$ and $d_j(x) > 0$, respectively, i.e.

$$J_0(x) = \{j : d_j(x) = 0\}, \qquad J_1(x) = \{j : d_j(x) > 0\}, j \in \overline{1, K}.$$

As a controlling process consider the process $\{U(t)\}_{t\geq 0}$, where $U(t)$ is a decision which should be taken at the next decision epoch. Let $A = \{0, 1, \dots, K\}$ be the set of available controls and

$$A(x) = \begin{cases} J_0(x) \cup \{0\}, & \text{for } x \text{ with } q(x) < B, \\ J_0(x), & \text{for } x \text{ with } q(x) = B, \end{cases}$$

be the set of admissible controls when the system state is x.
Suppose at a certain time instant t the system state $X(t) = x$. Then the controller chooses an admissible control $U(t) = a \in A(x)$, where $a = k \geq 1$ has the

meaning "switch on server k", whereas $a = 0$ has the meaning "send the job to the orbit".

Under the considered assumptions, the process $\{Z(t)\} = \{X(t), U(t)\}$ is a Markov decision one with finite state space E and finite control space $A(x) \subset A$ depending on the state $x \in E$.

Denote by $e_i = (\underbrace{0, \dots, 0}_{i}, 1, \underbrace{0, \dots, 0}_{K+1-i})$ the $K+2$ - dimensional vector for which the i-th component (beginning from 0-th) is one and all others are zeros. Consider the shift operators S_0, $S_j^{d_j}$ when arrivals of new customers take place in the queue

$$\begin{aligned} S_0 x &= x + e_0 \mathbf{1}_{\{q(x)<B\}}, \\ S_j^{d_j} x &= x + d_j e_j \mathbf{1}_{\{j \in J_0(x)\}}, \end{aligned}$$

otherwise, if $q(x) = B$ and $j \in J_1(x)$, $S_0 x = S_j^{d_j} x = x$. In case of exponential servers when a customer arrives the upper index of the operator S_j will be omitted. When a retrial arrival and a service completion take place one considers the inverse shift operators S_0^{-1} and $S_j^{-d_j}$ for the points $x \in E$, for which they exist, i.e.

$$\begin{aligned} S_0^{-1} x &= x - e_0 \mathbf{1}_{\{q(x)>0\}}, \\ S_j^{-d_j} x &= x - d_j e_j \mathbf{1}_{\{j \in J_1(x)\}}, \end{aligned}$$

otherwise, if $q(x) = 0$ and $j \in J_0(x)$, $S_0^{-1} x = x$ and $S_j^{-d_j} x = x$. The same operator $S_j^{d_j}$ is used to represent a phase change without service completion and modulating phase change

$$\begin{aligned} S_j^{d_j} x &= x + [d_j - d_j(x)] e_j \mathbf{1}_{\{j \in J_1(x)\}}, \\ S_{K+1}^{d_{K+1}} x &= x + [d_{K+1} - d_{K+1}(x)] e_{K+1}. \end{aligned}$$

Using the above notations we can represent the transition intensities $\lambda_{xy}(a)$ of the process $\{Z(t)\}$ to go from state x to state y, when action a is selected, in the form

$$\lambda_{xy}(a) = \begin{cases} \lambda_{d_{K+1}(x) d_{K+1}(y)}, & y = S_{K+1}^{d_{K+1}(y)} x, \\ \nu_{d_{K+1}(x) d_{K+1}(y)} [\mathbf{1}_{\{a=0\}} & \\ + \eta_a^{d_a(y)} \mathbf{1}_{\{a \neq 0\}}], & y = S_a^{d_a} S_{K+1}^{d_{K+1}(y)} x, \ a \in A(S_{K+1}^{d_{K+1}(y)} x), \\ \mu_j^{d_j(x)}, & y = S_j^{-d_j} x, \quad j \in J_1(x), \\ \mu_j^{d_j(x) d_j(y)}, & y = S_j^{d_j(y)} x, \quad j \in J_1(x) \cap J_1(y), \\ q(x) \gamma \eta_a^{d_a(y)} \mathbf{1}_{\{a \neq 0\}}, & y = S_a^{d_a} S_0^{-1} x, \ a \in A(S_0^{-1} x), \end{cases}$$

where $\mu_j^{d_j(x)}$ are the components of the vector $\vec{\mu}_j = -M_j\vec{1}$. The diagonal entries are the negative sums of the non–diagonal entries in the same row, and all positions not specified above are zero.
In this matrix

- the first row corresponds to a transition of an arrival phase without a new job arrival
- the second row corresponds to an arrival of a new job in the system and sending it in the orbit or to some server in accordance with the decision rule
- the third row corresponds to the service completion
- the fourth row corresponds to the service phase changing
- the fifth row corresponds to the retrial arrival and sending one of jobs from the orbit to one of available servers in accordance with decision rule.

3 Optimality of threshold policies

3.1 Problem statement

In the present section we consider the problem of a mean Number of Jobs Minimization (NJM-problem). The total number of jobs in the system equals the sum of the jobs in the orbit and under service. For the NJM-problem the quantity functional underlying the minimization problem takes the form

$$Y(t) = \int_0^t \left(D_0(u) + \sum_{1\le k\le K} \mathbf{1}_{\{D_k(u)>0\}} \right) du.$$

Let $l(x) = q(x) + \sum_{1\le k\le K} \mathbf{1}_{\{d_k(x)>0\}}$ denote the number of jobs in state x (which does not depend on the control a). This number represents the sum of jobs in the orbit plus the number of busy servers.

A strategy (or a policy) is a rule for choosing control actions $a \in A$. In general it may depend on the history of system states and may be randomized. As usual in Markov decision theory (see [7, 17, 25]), we define a strategy δ and a probability distribution $\mathbf{P}^{\delta}_{x_0}(.) = \mathbf{P}(.|X(0) = x_0, \delta)$ which is a measure on the set of the *trajectories* (the sequence of the states and controls during the observation period) of the process $\{Z(t)\}$, given an initial state x_0 and a strategy δ. Further, let $\mathbf{E}^{\delta}_{x_0}[.] = \mathbf{E}[.|X(0) = x_0, \delta]$ denotes the expectation with respect to this distribution. Then the problem of minimizing the long-run average number of jobs in the system can be represented as follows: Minimize

$$g(x_0; \delta) = \limsup_{t\to\infty} \frac{1}{t}\mathbf{E}^{\delta}_{x_0}[Y(t)] \tag{2}$$

with respect to all admissible strategies δ.

As it is well known (see, for example [7, 19, 22, 25]) for a Markov decision problem with respect to the long-run average minimization criterion an optimal strategy is a stationary Markov one, i.e. it is determined by the optimal policy $f = \{f(x) : x \in E\}$ which can be found from the optimality equation for the process as a minimizer of its right-hand side, and the gain $\inf_\delta g(x_0; \delta)$ exists and for ergodic Markov process it is independent of the initial state x_0; that is, there is a real number $g = \inf_\delta g(x_0; \delta)$.

To specify the optimality equation for the model let us denote by

$$V(x,\, t) = \inf_{\delta} \mathbf{E}^{\delta}_{x}[Y(t)]$$

the minimal expected total sojourn time of all customers in the system until time t. Then the obvious relation is

$$\lim_{t\to\infty} \frac{1}{t} V(x,\, t) = \inf_{\delta} \lim_{t\to\infty} \frac{1}{t}\mathbf{E}^{\delta}_{x}[Y(t)] = g$$

for all x. The last relation motivates the relation (for a proof see e.g. [6]) that a function $v : E \to \mathbb{R}$ exists such that for each $x \in E$

$$V(x,\, t) = tg + v(x) + o(1), \quad \text{for large } t.$$

In fact the function $v(x)$ indicates the transient effect of the initial state on the expected sojourn time of the customers under the given strategy. We often refer to $v(x)$ as the loss in state x.

For small time interval of length h according to common Markov process arguments the following equation can be obtained

$$\begin{aligned}
&V(x,\, t+h) = l(x)h + V(x,\, t) \\
&+ \left(\lambda_{d_{K+1}(x)d_{K+1}(x)} + \sum_{j\in J_1(x)} \mu_j^{d_j(x)d_j(x)} - q(x)\gamma\right) hV(x,\, t) \\
&+ \sum_{d_{K+1}=1}^{m_{K+1}} \nu_{d_{K+1}(x)d_{K+1}} h \min_{k\in A(S_{K+1}^{d_{K+1}}x)} \left\{ V(S_0 S_{K+1}^{d_{K+1}} x,\, t), \sum_{d_k=1}^{m_k} \eta_k^{d_k} V(S_k^{d_k} S_{K+1}^{d_{K+1}} x,\, t) \right\}
\end{aligned}$$

$$
\begin{aligned}
&+ q(x)\gamma h \min_{k\in A(S_0^{-1}x)} \left\{ V(x,\,t), \sum_{d_k=1}^{m_k} \eta_k^{d_k} V(S_k^{d_k} S_0^{-1} x,\,t) \right\} \\
&+ \sum_{\substack{d_{K+1}=1 \\ d_{K+1}\neq d_{K+1}(x)}}^{m_{K+1}} \lambda_{d_{K+1}(x)d_{K+1}} h\, V(S_{K+1}^{d_{K+1}} x,\,t) + \sum_{j\in J_1(x)} \sum_{\substack{d_j=1 \\ d_j\neq d_j(x)}}^{m_j} \mu_j^{d_j(x)d_j} h\, V(S_j^{d_j} x,\,t) \\
&+ \sum_{j\in J_1(x)} \mu_j^{d_j} h\, V(S_j^{-d_j} x,\,t).
\end{aligned}
$$

The first term on the right hand side represents the sojourn time of $l(x)$ customers resident in the system during a time interval of duration h, the second term represents the total sojourn time of all customers being in the system during the subsequent time interval of duration t in case that there are no state changes, the next term represents the total sojourn time of all customers being in the system during time t in case that a new customer arrives before the next retrial customer arrives. The following term represents the total sojourn time of the customers in the system in case of a retrial arrival before a new customer arrival, the next two terms deal with the total sojourn time in the system in case of a phase change without arrival or service completion, and the remaining term represents the total sojourn time in the system during time t in case that one of the serviced customers leaves the system before some customer arrives.

After some elementary algebra, and passing to the limit $h \to 0$ the above equation leads to a differential optimality equation. Now by substituting the above asymptotic expansion for $t \to \infty$ in the differential equation, after canceling out common terms, the optimality equation assumes the following form

$$
\begin{aligned}
v(x) &= \frac{1}{\lambda_x} \Bigg[l(x) + C_1(x) + C_2(x) - g + \\
&+ \sum_{d_{K+1}=1}^{m_{K+1}} \nu_{d_{K+1}(x)d_{K+1}} Tv(S_{K+1}^{d_{K+1}} x) + \mathbf{1}_{\{a\in A(S_0^{-1}x)\setminus\{0\}\}} q(x)\gamma Tv(S_0^{-1}x) \Bigg] \\
&= Bv(x),
\end{aligned}
\tag{3}
$$

where $Bv(x)$ denotes the transform operator for the function $v(x)$, (see [21]). In this representation

$$
\lambda_x = -\left(\lambda_{d_{K+1}(x)d_{K+1}(x)} + \sum_{j\in J_1(x)} \mu_j^{d_j(x)d_j(x)} - \mathbf{1}_{\{a\in A(S_0^{-1}x)\setminus\{0\}\}} q(x)\gamma \right)
$$

is the total transition intensity out of state x;

$$C_1(x) = \sum_{d_{K+1}\neq d_{K+1}(x)} \lambda_{d_{K+1}(x)d_{K+1}} v(S_{K+1}^{d_{K+1}}x) + \sum_{j\in J_1(x)} \sum_{d_j\neq d_j(x)} \mu_j^{d_j(x)d_j} v(S_j^{d_j}x)$$

is called the loss rate due to a transition of the MAP or the service phase without decision making;

$$C_2(x) = \sum_{j\in J_1(x)} \mu_j^{d_j(x)} v(S_j^{-d_j}x)$$

is the loss rate due to a service completion;

$$Tv(x) = \min\left\{ v(S_0x), \sum_{d_k=1}^{m_k} \eta_k^{d_k} v(S_k^{d_k}x) : \; k = \overline{1,K} \right\} \tag{4}$$

is called the minimal loss in the case of a new arrival to state x. The same operator at the point $S_0^{-1}x$ represents the minimal loss in case of a retrial job.

As in case of a classic queue (see [5, 21]) the form of the optimality equation shows that the optimal policy $f = \{f(x) : x \in E\}$ is completely determined by the value function $v = \{v(x) : x \in E\}$ which in turn is a solution of the optimality equation (3), namely

$$f(x) = \operatorname{argmin} \begin{cases} v(S_ax), & \text{exponential service}, \\ \mathbf{1}_{\{a=0\}} v(S_0x) + \mathbf{1}_{\{a=k>0\}} v(S_k^1x), & \text{Erlangian service}, \\ \mathbf{1}_{\{a=0\}} v(S_0x) + \mathbf{1}_{\{a=k>0\}} \sum_{d_k=1}^{m_k} \eta_k^{d_k} v(S_k^{d_k}x), & \text{PH-type service}. \end{cases} \tag{5}$$

Thus the function $f(x)$ specifies the optimal decision rule which has to be taken in case of a new or retrial request's arrival in the state x.

3.2 Assignment to the fastest available server

Our objective in this section is to prove that it is optimal to use the fastest available server. As shown in the previous section the optimal policy is completely determined by the value function $v(x)$, therefore, it is necessary to investigate the properties of this function. The form of the optimality equation (3) shows that for retrial queueing systems all the properties of the function $v(x)$ for ordinary queues also should hold in this case, see for details [5]. Since these assertions can only be partially proved, they are formulated as conjectures.

Conjecture 1: *The value function of the model* $v = \{v(x) : \ x \in E\}$ *satisfies the following monotonicity properties*

$$
\begin{aligned}
&1. \quad \sum_{d_j=1}^{m_j} \eta_j^{d_j} v(S_j^{d_j} x) \leq \sum_{d_i=1}^{m_i} \eta_i^{d_i} v(S_i^{d_i} x), i, j \in J_0(x), \quad \overline{\mu_j}^{-1} \leq \overline{\mu_i}^{-1}, \\
&2. \quad \sum_{d_1=1}^{m_1} \eta_1^{d_1} v(S_1^{d_1} x) \leq v(S_0 x), \\
&3. \quad v(x) \leq v(S_0 x), v(x) \leq v(S_i^{d_i} x), \quad d_i = \overline{1, m_i}, \\
&4. \quad v(S_i^{\alpha} x) \leq v(S_i^{\beta} x), \quad i \in J_0(x), \mu_i^{\alpha} \geq \mu_i^{\beta}, \\
&5. \quad v(S_{K+1}^{\alpha} x) \leq v(S_{K+1}^{\beta} x), \nu_{\alpha} \leq \nu_{\beta}.
\end{aligned}
$$

Property 1 means that the controller has to activate only the fastest server available in state x. According to property 2, whenever the fastest (the first) server is idle it is always optimal to allocate a job to this server. The next property 3 describes the monotonicity condition of the value function with respect to the shifts S_0 and $S_i^{d_i}$. The last two properties 4 and 5, where $\mu_i^{d_i}$ is the total service rate from phase d_i and $\nu_{d_{K+1}}$ denotes the total arrival rate from the modulating phase d_{K+1}, show the monotonicity of the value function with respect to the different states of arrival and service processes.

To prove these inequalities it is necessary to show that the operators B and T, introduced by (3) and (4), respectively, preserve them for the function which is monotone with respect to the partial order introduced on the state space E, see [5, 21]. Then the inequalities follow from the monotone convergence $\lim\limits_{n\to\infty} B^n l(x) = v(x)$, as shown in Howard [6], and the fact that the function $l(x)$ preserves the mentioned monotonicity properties.
A rigorous proof of these inequalities has been pursued only for simplified queues (e.g. with exponential servers) and is analogous to a proof for ordinary queueing systems, as investigated in [5].

As an example we prove the first inequality of the Conjecture 1 for the M/M/K retrial queue.

Proof: To study the monotonicity properties of the value function we exploit a partial ordering of the state space E, and the complete ordering of the set A of controls. For that purpose the servers are arranged in the order of decreasing service intensities (increasing mean service times)

$$0 \le \mu_1^{-1} \le \mu_2^{-1} \le \cdots \le \mu_K^{-1} \tag{6}$$

and the components of the vector $d = (d_1, \dots, d_K)$ are numbered accordingly. Assume, that the operators S_0 and S_i shift the points of E in positive direction, that is,

$$S_0 x \ge x \quad \text{and} \quad S_i x \ge x, \quad i \in J_0(x). \tag{7}$$

Shifted points are ordered with respect to increasing mean service times, i.e.

$$S_i x \ge S_j x, \quad \text{if } i \ge j, \quad i, j \in J_0(x) \quad (\text{that is } \mu_i^{-1} \ge \mu_j^{-1}). \tag{8}$$

Points $S_0 x$ and $S_j x$, $(j \neq 0)$ are not comparable.

In the set A of controls a complete ordering is given according to the numbering $1 < 2 < \cdots < K$ (with respect to servers $1, \dots, K$). Clearly, this induces the corresponding ordering in any subset $A(x)$.

To prove e.g. the first property in Conjecture 1 for the queueing system under consideration we have to prove that the following is true for each value $x \in E$

$$Tv(S_i x) \ge Tv(x),\, i \in A(x), \quad Tv(S_i x) \ge Tv(S_j x),\, i, j \in J_0(x),\, i \ge j.$$

Let the function $v(x)$ be nondecreasing with respect to the introduced ordering, namely

$$v(S_i x) \ge v(x), \quad i \in A(x); \qquad v(S_i x) \ge v(S_j x),\, i, j \in J_0(x),\, i \ge j.$$

For the first inequality we have for $i \in A(x)$

$$Tv(S_i x) = \min_{k \in A(S_i x)} v(S_k S_i x) \ge \min_{k \in A(S_i x)} v(S_k x) \ge \min_{k \in A(x)} v(S_k x) = Tv(x),$$

where the first item follows from assumption that the function $v(x)$ is nondecreasing and the second item follows from relation $A(S_i x) \subset A(x)$ together with the fact that the minimum does not increase upon expanding the minimization set.

Finally, we prove that upon passing from S_j to S_i, $i, j \in J_0(x)$, $i \ge j$, the operator T_0 preserves the property to be nondecreasing. Let $i, j \in J_0(x)$. Then the set $A(x)$ can be represented as $A(x) = B(x) \cup \{i\} \cup \{j\}$ with some $B(x)$ so that $A(S_i x) = B \cup \{j\}$, $A(S_j x) = B(x) \cup \{i\}$ and the relations

$$Tv(S_i x) = \min_{k \in A(S_i x)} v(S_k S_i x) =$$

$$\min\left\{ \min_{k \in B(x)} v(S_k S_i x), v(S_i S_j x) \right\}$$

$$\geq \min\left\{ \min_{k \in B(x)} v(S_k S_j x), v(S_i S_j x) \right\} = Tv(S_j x)$$

are valid by virtue of inequality $v(S_i x) \geq v(S_j x)$.

Now we show that the operator B defined by (3) also retains monotonicity upon passing from $S_j x$ to $S_i x$, $i, j \in J_0(x),\ i \geq j$.
Let us multiply each side of equation (3) for the state $S_i x$ by $\lambda_{S_i x}$ and add

$$(\mu_j + \mathbf{1}_{\{a \in A(S_0^{-1} S_j x) \setminus \{0\}\}} q(S_j x)\gamma) v(S_i x)$$

to each side.
Analogously, for the state $S_j x$ we multiply the optimality equation by $\lambda_{S_j x}$ and add

$$(\mu_i + \mathbf{1}_{\{a \in A(S_0^{-1} S_i x) \setminus \{0\}\}} q(S_i x)\gamma) v(S_j x)$$

to each side.
To simplify the form of an optimality equation we make the following notation

$$(\lambda_{S_i x} + \mu_j + \mathbf{1}_{\{a \in A(S_0^{-1} S_j x) \setminus \{0\}\}} q(S_j x)\gamma)) = (\lambda_{S_j x} + \mu_i + \mathbf{1}_{\{a \in A(S_0^{-1} S_i x) \setminus \{0\}\}} q(S_i x)\gamma)) = w.$$

Finally, for nondecreasing function $v(x)$ for any $i,\ j \in J_0(x)$ such that $i \geq j$, taking into account that $q(S_i x) = q(S_j x) = q(x)$, we get

$$\begin{aligned} Bv(S_i x) - Bv(S_j x) = \frac{1}{w}\Bigg(& [l(S_i x) - l(S_j x)] + \lambda[Tv(S_i x) - Tv(S_j x)] \\ & + \sum_{l \in J_1(x)} \mu_l [v(S_i S_l^{-1} x) - v(S_j S_l^{-1} x)] \\ & + \mu_j [v(S_i x) - v(x)] - \mu_i [v(S_j x) - v(x)] \\ & + \mathbf{1}_{\{a \in A(S_0^{-1} S_i x) \setminus \{0\}\}} q(x)\gamma [Tv(S_0^{-1} S_i x) - v(S_j x)] \\ & - \mathbf{1}_{\{a \in A(S_0^{-1} S_j x) \setminus \{0\}\}} q(x)\gamma [Tv(S_0^{-1} S_j x) - v(S_i x)] \Bigg) \geq 0, \end{aligned}$$

where the first three items in the right-hand side are nonnegative by virtue of the constancy of the function $l(x)$ upon passing from S_jx to $S_ix,\ i,j \in J_0(x),\ i \geq j$ and the fact that operator T retains the monotonicity of the functions.
For the next item we obtain

$$\mu_j[v(S_ix) - v(x)] - \mu_i[v(S_jx) - v(x)] =$$
$$\mu_i\mu_j\left[\frac{v(S_ix) - v(x)}{\mu_i} - \frac{v(S_jx) - v(x)}{\mu_j}\right] \geq 0,$$

since

$$v(S_ix) - v(x) \geq v(S_jx) - v(x) \geq 0$$

and $\mu_i \leq \mu_j$, owing to the monotonicity assumption.
For the last item we consider several subcases. If a control $a = 0$ in states $S_0^{-1}S_ix$ and $S_0^{-1}S_jx$ then this item is equal to zero. In case $a \neq 0$ in both of states we get

$$q(x)\gamma[[Tv(S_0^{-1}S_ix) + v(S_ix)] - [Tv(S_0^{-1}S_jx) + v(S_jx)]] \geq 0$$

owing to the monotonicity assumption and property of the operator T. In case if $a \neq 0$ in state $S_0^{-1}S_ix$ and $a = 0$ in state $S_0^{-1}S_jx$ we obtain

$$q(x)\gamma[Tv(S_0^{-1}S_ix) - v(S_jx)] \geq 0,$$

since in each state the decision has been made if necessary and optimality equation (3) is defined only for the sTable states x. It means that this function satisfies a condition

$$v(S_jx) = \min_{k\in A(S_0^{-1}S_jx)} v(S_kS_0^{-1}S_jx) \leq \min_{k\in A(S_0^{-1}S_ix)\setminus\{0\}} v(S_kS_0^{-1}S_ix) = Tv(S_0^{-1}S_ix).$$

Finally, if $a = 0$ in state $S_0^{-1}S_ix$ and $a \neq 0$ in state $S_0^{-1}S_jx$ then analogously we get

$$q(x)\gamma[Tv(S_0^{-1}S_jx) - v(S_ix)] \leq 0,$$

due to the properties of the function $v(x)$

$$v(S_ix) \geq v(S_jx) \geq \min_{k\in A(S_0^{-1}S_jx)\setminus\{0\}} v(S_kS_0^{-1}S_jx) = Tv(S_0^{-1}S_jx).$$

It should be noted that the operator B retains inequalities in the sense that if $v(x) \geq v(y)$ then $Bv(x) \geq Bv(y)$. Therefore the theorem follows from the mentioned above monotonicity convergence of the sequence $B^nl(x)$ to the value $v(x)$ and the fact, that the function $l(x)$ is constant relative to shifts from S_i to S_j.

3.3 The optimal policy is of threshold type

Based on the properties of the value function $v(x)$ the following strengthened conjecture can be formulated:

Conjecture 2: *Under the conditions of Conjecture 1 the value function $v = \{v(x) : x \in E\}$ of the model satisfies the following monotonicity properties of the increments:*

1. $v(S_0x) - \sum_{d_k=1}^{m_k} \eta_k^{d_k} v(S_k^{d_k}x) \le v(S_0^2x) - \sum_{d_k=1}^{m_k} \eta_k^{d_k} v(S_k^{d_k}S_0x),$

2. $v(S_0S_i^{\alpha}x) - \sum_{d_k=1}^{m_k} \eta_k^{d_k} v(S_k^{d_k}S_i^{\alpha}x) \le v(S_0S_i^{\beta}x) - \sum_{d_k=1}^{m_k} \eta_k^{d_k} v(S_k^{d_k}S_i^{\beta}x),$

   $\mu_i^{\alpha} \ge \mu_i^{\beta},$

3. $v(S_0S_{K+1}^{\alpha}x) - \sum_{d_k=1}^{m_k} \eta_k^{d_k} v(S_k^{d_k}S_{K+1}^{\alpha}x) \le v(S_0S_{K+1}^{\beta}x) - \sum_{d_k=1}^{m_k} \eta_k^{d_k} v(S_k^{d_k}S_{K+1}^{\beta}x),$

   $\nu_{\alpha} \le \nu_{\beta}.$

Property 1 of this conjecture means that if in some state it is optimal to keep a customer in the orbit, then this control action is also optimal in all states with the same collection of busy servers and less number of jobs being in the orbit. Properties 2 and 3 show that the incentive to make an assignment to the k-th slower server is greater in the state $S_{K+1}^{\beta}x$ with larger arrival intensity $\nu_{\beta} \ge \nu_{\alpha}$ and smaller service intensity $\mu_i^{\beta} \le \mu_i^{\alpha}$ of some i-th faster server. These inequalities can be proved in the same way as in [5, 21].

As before we consider the simplified queue M/M/K to prove the first assertion in Conjecture 2.

Proof: Regarding to the servers switching rule we show that an optimal policy has a threshold type, i.e. if in some state $x \in E$ with $q(x)$ jobs in the orbit at the decision epoch it is optimal to leave a job in the orbit then this control action will be optimal in all states y with the same collection of busy servers and $q(y) \le q(x)$. In other words, the equality $f(x) = 0$ leads to the equality $f(y) = 0$ if $q(y) \le q(x)$. For this it is sufficient that

$$f(S_0x) = 0 \quad \Rightarrow \quad f(x) = 0.$$

It is possible to rewrite the last relation in accordance with the definition of optimal policy (5) in the form

$$v(S_0^2x) \geq v(S_0S_kx) \Rightarrow v(S_0x) \geq v(S_kx) \quad \text{for all } k \in J_0(x)$$

or as inequality

$$v(S_0x) - v(S_kx) - v(S_0^2x) + v(S_0S_kx) \leq 0. \tag{9}$$

According to the inequality 1 in Conjecture 1 only two solutions are possible in each state x: $f(x) = 0$ (not to serve the job) or $f(x) = k$ (to use the fastest free server), so that here the family $A(x)$ of controls is independent on the shift S_0. We set out to prove that the operator T retains the above inequality. Thus it is necessary to check whether this property is satisfied for the function $\hat{v}(x) = Tv(x)$ if it is satisfied for some function $v(x)$. That is we have to prove

$$\begin{aligned}
&\hat{v}(S_0x) - \hat{v}(S_kx) - \hat{v}(S_0^2x) + \hat{v}(S_0S_kx) = \\
&\min\{v(S_lS_0x) : l \in J_0(x)\} - \min\{v(S_lS_kx) : l \in J_0(S_kx)\} - \\
&\min\{v(S_lS_0^2x) : l \in J_0(x)\} + \min\{v(S_lS_0S_kx) : l \in J_0(S_kx)\} \leq 0.
\end{aligned}$$

To prove this assertion for each point $x \in E$ we divide it into several cases

1. First we consider the case when the optimal solutions coincide at the points S_kx and S_0^2x (where $\hat{v}(x)$ is involved in inequality with negative sign), and $f(S_kx) = f(S_0^2x) = f$. Obviously, by replacing the optimal solution at the rest of the points by f, we obtain that

$$\begin{aligned}
&\hat{v}(S_0x) - \hat{v}(S_kx) - \hat{v}(S_0^2x) + \hat{v}(S_0S_kx) = \\
&\hat{v}(S_0x) - v(S_fS_kx) - v(S_fS_0^2x) + \hat{v}(S_0S_kx) \leq \\
&v(S_fS_0x) - v(S_fS_kx) - v(S_fS_0^2x) + v(S_fS_0S_kx) \leq 0.
\end{aligned}$$

2. The case of different optimal solutions at the points S_kx and S_0^2x (where $\hat{v}(x)$ is involved in inequality with negative sign) should be divided into two subcases: $f(S_0^2x) = 0,\ f(S_kx) = l \neq 0$, where l is the index of the fastest available server, and $f(S_0^2x) = k \neq 0,\ f(S_kx) = 0$.

In the first subcase, by summing the inequalities (9) at the points S_kx and S_0x, respectively,

$$v(S_0S_kx) - v(S_lS_kx) - v(S_0^2S_kx) + v(S_0S_lS_kx) \leq 0,$$

and

$$v(S_0^2x) - v(S_0S_kx) - v(S_0^3x) + v(S_0^2S_kx) \le 0,$$

we get the inequality

$$v(S_0^2x) - v(S_kS_lx) - v(S_0^3x) + v(S_kS_lS_0x) \le 0,$$

which shows that the inequality (9) is satisfied for the function $\hat{v}(x)$ at the point x since

$$\begin{aligned}
&\hat{v}(S_0x) - \hat{v}(S_kx) - \hat{v}(S_0^2x) + \hat{v}(S_kS_0x) = \\
&v(S_0^2x) - v(S_kS_lx) - v(S_0^3x) + v(S_kS_lS_0x) \le 0.
\end{aligned}$$

In the second subcase, the relation has the form

$$\begin{aligned}
&\hat{v}(S_0x) - \hat{v}(S_kx) - \hat{v}(S_0^2x) + \hat{v}(S_0S_kx) = \\
&\hat{v}(S_0^2x) - v(S_0S_kx) - v(S_kS_0^2x) + \hat{v}(S_0S_kx) \le \\
&v(S_kS_0x) - v(S_0S_kx) - v(S_kS_0^2x) + v(S_0S_0S_kx) = 0.
\end{aligned}$$

For the boundary points $q(x) = B,\ J_0(x) = \emptyset$, the inequality (9) for the function $\hat{v}(x)$ is also satisfied due to the definition of the shift operators. Now the first assertion of Conjecture 2 follows from the fact that property of (9) is retained for linear operations defining the operator B, the function $l(x)$ satisfies (9) and the successive approximations $B^n l(x)$ converge monotonously to the value function $v(x)$.

Thus for some simplified retrial queues we can prove and for general system we may expect that the optimal policy is of threshold type, as stated below:
Corollary: *The optimal policy for the controlled retrial system $MAP/PH/K$ is of threshold type with finite thresholds $q_j^*(d_1, \ldots, d_K, d_{K+1}),\ d_i > 0,\ i = \overline{1, j-1}$, for each arrival and service phase and it is necessary to switch on the j-th server only if $q(x) \ge q_j^*$. In case of the NJM-problem the decision maker has to use the fastest available server.*

It is obvious that if the orbit is nonempty and the first server with mean service time $\bar{\mu}_1^{-1} = \min\limits_{k \in J_0(x)} \{\bar{\mu}_k^{-1}\}$ is available (idle), then the idleness of the fastest server is never optimal, i.e. the threshold level for the first server $q_1^* = 0$.
We note that for the NJM-problem the threshold level for the j-th server $q_j^*(d_1, \ldots, d_K, d_{K+1})$ depends on the states of arrival and service processes, i.e. it can

depend on the states of slower servers. Numerical results show that such influence may arise only in case of a large arrival intensity and the threshold levels can vary by at most 1 when the states of slower servers change.

Now assume that in our system there are n jobs and no future arrivals take place. The jobs must be served as soon as possible. This problem is known as "scheduling problem".

In case $K = 2$ servers it is possible to obtain the threshold level explicitly. Indeed, in this case $g = 0$. Now using the equation (3), taking into account the above statement, we just want to clear a system which already contains the customers. Solving recursively the optimality equation we can find the threshold levels $q_2^*(d_1)$ for the second server. By virtue of the threshold property of the optimal policy if the finite state $x = (0, 0, 0)$ we get

$$\begin{pmatrix} v(S_0^{q_2^*(1)-1} S_1^1 x) \\ v(S_0^{q_2^*(2)-1} S_1^2 x) \\ \cdot \\ \cdot \\ \cdot \\ v(S_0^{q_2^*(m_1)-1} S_1^{m_1} x) \end{pmatrix} = -M_1^{-1}\vec{\mathbf{1}} q_2^* + \vec{\mathbf{1}} \left[\frac{q_2^*(q_2^*+1)}{2\overline{\mu}_1} + \frac{(q_2^*-1)}{\gamma} \right].$$

If $q_2^*(d_1)$ is a threshold for using the second server the following inequality holds

$$\sum_{d_2=1}^{m_2} \eta_2^{d_2} v(S_0^{q_2^*(d_1)-1} S_1^{d_1} S_2^{d_2} x) \le v(S_0^{q_2^*(d_1)} S_1^{d_1} x)$$

For the last inequality we get

$$\sum_{d_2=1}^{m_2} \eta_2^{d_2} v(S_0^{q_2^*(d_1)-1} S_1^{d_1} S_2^{d_2} x) = \vec{\mathbf{1}} \frac{1}{\overline{\mu}_2} + v(S_0^{q_2^*(d_1)-1} S_1^{d_1} x) \le v(S_0^{q_2^*(d_1)} S_1^{d_1} x)$$

and now we obtain the vector of thresholds for the second server for each service phase

$$\begin{pmatrix} q_2(1) \\ q_2(2) \\ \cdot \\ \cdot \\ \cdot \\ q_2(m_1) \end{pmatrix} \le \begin{pmatrix} q_2^*(1) \\ q_2^*(2) \\ \cdot \\ \cdot \\ \cdot \\ q_2^*(m_1) \end{pmatrix} = \left\lfloor \vec{\mathbf{1}} \left(\frac{\overline{\mu}_1}{\overline{\mu}_2} - \frac{\overline{\mu}_1}{\gamma} \right) + M_1^{-1} \vec{\mathbf{1}} \overline{\mu}_1 \right\rfloor.$$

For the value of "mean" threshold we have

$$\overline{q_2^*} = \left\lfloor \sum_{d_1=1}^{m_1} \eta_1^{d_1} q_2^*(d_1) \right\rfloor = \left\lfloor \left(\frac{1}{\overline{\mu}_2} - \frac{1}{\gamma} \right) \overline{\mu}_1 - 1 \right\rfloor.$$

In case $K > 2$ it can be shown that for the threshold levels

$$\overline{q_j^*} = \left\lfloor \sum_{d_1=1}^{m_1} \cdots \sum_{d_{j-1}=1}^{m_{j-1}} \eta_1^{d_1} \dots \eta_{j-1}^{d_{j-1}} q_j^*(d_1, \dots, d_{j-1}) \right\rfloor, \qquad j = 2, \dots, K$$

the bounds can be obtained, namely

$$\left\lfloor \left(\frac{1}{\overline{\mu}_j} - \frac{1}{\gamma} \right) \sum_{k=1}^{j-1} \overline{\mu}_k - (j-1) \right\rfloor \le \overline{q_j^*} \le \left\lfloor \frac{1}{\overline{\mu}_j} \sum_{i=1}^{j-1} \overline{\mu}_i - (j-1) \right\rfloor.$$

We expect that this values represent the bounds for the threshold levels also in a so-called light traffic case, when $\overline{\lambda} \in [0, \overline{\mu}_K)$. Finally, when the retrial intensity γ is large then the model turns to be the classical queueing model. In this case the bounds for the threshold levels coincide.

4 An Algorithm

The following algorithm is based on Howard's iteration algorithm [6] but it has been modified with respect to specific properties of the problem. The algorithm consists of two basic steps: *Value function evaluation* and *Policy improvement*.

Value function evaluation. For a given policy $f = \{f_n(x) : x = \overline{0, I}\}$, where I is defined below, starting from $n = 0$ solve the equation (3) by a successive approximation method with given accuracy ε

$$v_n(x) = \frac{1}{\lambda_x} \Big(l(x) + C_1(x) + C_2(x) - g_n \Big) +$$

$$\frac{1}{\lambda_x} \sum_{d_{K+1}=1}^{m_{K+1}} \nu_{d_{K+1}(x) d_{K+1}}$$

$$\left(\mathbf{1}_{\{f_n(S_{K+1}^{d_{K+1}} x)=0\}} v_n(S_0 S_{K+1}^{d_{K+1}} x) + \mathbf{1}_{\{f_n(S_{K+1}^{d_{K+1}} x)=k\}} \sum_{d_k=1}^{m_k} \eta_k^{d_k} v_n(S_k^{d_k} S_{K+1}^{d_{K+1}} x) \right) +$$

$$\frac{1}{\lambda_x} \mathbf{1}_{\{f_n(S_0^{-1} x)=k \neq 0\}} \sum_{d_k=1}^{m_k} \eta_k^{d_k} v_n(S_k^{d_k} S_0^{-1} x)$$

for all $x \in E$ under the condition $v_n(0) = 0$.

Policy improvement. For a given solution $v_n = \{v_n(x) : x \in E\}$ find a new policy $f_{n+1} = \{f_{n+1}(x) : x \in E\}$ (5):

$$f_{n+1}(x) = \operatorname*{argmin}_{k \in A(x)} \begin{cases} \sum_{d_k=1}^{m_k} \eta_k^{d_k} v_n(x + d_k e_k), & k = \overline{1, K}, \\ v_n(x + e_0), & k = 0. \end{cases}$$

The algorithm stops when two successive iterations yield the same policy.

To describe the system state changes we consider the one-to-one correspondence between the multi-dimensional representation of the system state x and the index of such a state. Namely,

$$\#(x) = \prod_{i=1}^{K}(m_i+1)(d_0(x)m_{K+1}+d_{K+1}(x)-1)+\sum_{j=1}^{K} d_j(x)\mathbf{1}_{\{j>1\}}\prod_{i=1}^{j-1}(m_i+1) \equiv x,$$

where $x = \overline{0, I},\ I = \prod_{i=1}^{K}(m_i + 1)m_{K+1}(B + 1) - 1$.

Now, if y_j is the state after a possible transition from the j-th coordinate it can be obtained with respect to the formula

$$y_j = x + \frac{(d_j - d_j(x)) \prod_{i=1}^{K}(m_i + 1)\mathbf{1}_{\{j=0\}} m_{K+1}}{\mathbf{1}_{\{1\le j\le K\}} \prod_{i=j}^{K}(m_i + 1)}.$$

Thus, in the one-dimensional case we have

$$S_0 x = x + m_{K+1} \prod_{i=1}^{K}(m_i + 1),$$

$$S_0^{-1} x = x - m_{K+1} \prod_{i=1}^{K}(m_i + 1),$$

$$S_j^{d_j} x = x + (d_j - d_j(x)) \prod_{i=1}^{j-1}(m_i + 1),$$

$$S_j^{-d_j} x = x - d_j(x) \prod_{i=1}^{j-1}(m_i + 1),$$

$$S_{K+1}^{d_{K+1}} = x + (d_{K+1} - d_{K+1}(x)) \prod_{i=1}^{K}(m_i + 1).$$

Using this algorithm one can construct explicit forms of optimal policies for the retrial queueing system $MAP/PH_{het}/K$ and any particular case of this system.

5 Numerical Examples

For numerical results of optimal policies we will consider the retrial queueing systems $M/M/K,\ MAP/M/K,\ M/E/K$ and $MAP/E/K$ and investigate the results for different system parameters. The numerical analysis is based on a series of experiments. The optimal control policies will be represented by means of *control tables* and *control diagrams*. Before we discuss the obtained results we give some necessary comments. In all these examples the servers are arranged in increasing order of their mean service times (1) and the queueing systems are considered with $B = 100$.

The **control tables** show the optimal control policies $f(x)$ for each system state in case of $K = 5$ heterogeneous servers when the values of system parameters are fixed. The left column represents the list of serverst' states and the upper row represents the number of jobs in the orbit. The optimal control actions that correspond to the threshold levels (critical number of jobs in the orbit) are underlined. As it was mentioned above, the optimal decision rule for using some server j shows a weak dependence on the states of slower servers, therefore the states of slower servers are labeled as "*", that means 0 or $d_i > 0$ for $i > j$.

By means of **control diagrams** we investigate the queues with $K = 3$ servers. The diagrams show the behaviour of threshold functions q_j^* when the values of system parameters are varied. In this section we investigate the threshold function q_2^* for the second server with varying service intensities of the first two servers, i.e. the thresholds which occur in the states $x = (d_1, 0, *)$, $d_1 > 0$. In fact, the threshold function q_3^* for the third server in the states $x = (d_1, d_2, 0)$, $d_1, d_2 > 0$, have the same structure and illustrate the same properties as for the second server, but depending on the service intensities of all three servers. The service intensity for the third server in control diagrams for all systems is fixed, i.e. $\mu_3 = 0.05$.

5.1 $M/M/K$ system

The following Table gives the optimal policies for the system with $K = 5$ servers and parameter values $\lambda = 0.01, \mu_i = \{2.50, 0.63, 0.52, 0.40, 0.30\},\ \gamma = 2.90$.

Table 1: Optimal control in each system state

System State x	Number of jobs in the orbit $q(x)$													
$(d_1, d_2, d_3, d_4, d_5)$	0	1	2	3	4	5	6	7	8	9	10	11	12	$\dots$
(0,*,*,*,*)	**1**	1	1	1	1	1	1	1	1	1	1	1	1	1
(1,0,*,*,*)	0	0	**2**	2	2	2	2	2	2	2	2	2	2	2
(1,1,0,*,*)	0	0	0	**3**	3	3	3	3	3	3	3	3	3	3
(1,1,1,0,*)	0	0	0	0	0	**4**	4	4	4	4	4	4	4	4
(1,1,1,1,0)	0	0	0	0	0	0	0	0	**5**	5	5	5	5	5
(1,1,1,1,1)	0	0	0	0	0	0	0	0	0	0	0	0	0	0

In the Tables the optimal control actions which correspond to the threshold levels are underlined. The Table shows that while the fastest server is always activated, each slower server has a threshold that prescribes when to switch on this server. Therefore, the optimal control rule can be described by means of the threshold sequence $0 = q_1^* \leq q_2^* \leq \cdots \leq q_K^*$, where $q_1^* = 0$, $q_2^* = 2$, $q_3^* = 3$, $q_4^* = 5$ and q_5^*=8.

Some results for this system are summarized in the diagrams, shown in the Figures 1.1(a,b)-1.3(a,b) in Appendix 1. In these diagrams the changing of the threshold levels q_2^* for second server represents the threshold function under the variation of the first service intensity for different values of the second service intensity.

Different types of curves are used to show the threshold levels behavior for different values of the second service intensity. The legend of the diagrams represents the states where threshold levels occur, i.e. for the second server the states are $x = (1, 0, *)$.

The retrial intensity γ and input intensity λ are varied over the Figures:

- γ=0.3 in Fig.1.1,
- γ=0.5 in Fig.1.2,
- γ=0.9 in Fig.1.3,
- λ=0.01 (pictures labeled by letter "b").
- λ=0.51 (pictures labeled by letter "a"),

From these diagrams one can see that the curves have a step structure which shows the threshold phenomenon of the optimal policies for the model under consideration. The threshold behavior depends on service intensities, and the threshold levels for the slower second server monotically increases when the first service intensity increases, and/or the second service intensity decreases. The retrial and arrival intensities also influence the threshold behavior. Namely, the threshold level decreases when the arrival intensity increases, and/or the retrial intensity decreases.

5.2 $MAP/M/K$ system

The following Table shows that the optimal thresholds can depend on the modulating state. The parameter values are the same as before with mean arrival rate $\overline{\lambda} = 0.01$. We select elements of the matrix N of dimension $m_{K+1} = 5$ such that $\nu_\alpha \leq \nu_\beta,\ \alpha \geq \beta$.

Table 2: Optimal control in each system state

System State x	Number of jobs in the orbit $q(x)$													
$(d_1, d_2, d_3, d_4, d_5, d_6)$	0	1	2	3	4	5	6	7	8	9	10	11	12	...
(0,*,*,*,*,*)	**1**	1	1	1	1	1	1	1	1	1	1	1	1	1
(1,0,*,*,*,*)	0	0	**2**	2	2	2	2	2	2	2	2	2	2	2
(1,1,0,*,*,*)	0	0	0	**3**	3	3	3	3	3	3	3	3	3	3
(1,1,1,*,*,1)	0	0	0	0	0	**4**	4	4	4	4	4	4	4	4
(1,1,1,*,*,5)	0	0	0	0	0	0	**4**	4	4	4	4	4	4	4
(1,1,1,1,0,*)	0	0	0	0	0	0	0	0	**5**	5	5	5	5	5
(1,1,1,1,1,*)	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Now the threshold sequence $0 = q_1^* \leq q_2^*(d_{K+1}) \leq \cdots \leq q_K^*(d_{K+1})$ depends on the state of the arrival process. In this example $q_k^*(\alpha) \geq q_k^*(\beta)$ if $\nu_\alpha \leq \nu_\beta$.
The influence of the retrial intensity on the threshold behavior for this model is illustrated in Appendix 2 in Figure 2.1:

- γ=0.9 (pictures labeled by letter "b").
- γ=0.3 (pictures labeled by letter "a"),

The rate matrices for MAP are the following

$$N = \begin{pmatrix} 0.00 & 0.34 \\ 1.00 & 0.00 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} -0.34 & 0.00 \\ 0.00 & -1.00 \end{pmatrix},$$

with average arrival rate $\overline{\lambda} = 0.51$.

Different types of curves are also used to show the behavior of the threshold levels for varied service intensities but now for different system states with respect to the MAP. These diagrams show an analogous behavior of threshold levels as in the previous system when the intensity γ changes, i.e. threshold levels increase if the retrial intensity γ increases. But now it is possible to see that thresholds in these examples depend also on the states of the MAP. Such an influence is illustrated in the examples shown in Figure 3.1 of Appendix 3. We investigated the system with different rate matrices for the MAP:

- Fig.3.1 a

$$N = \begin{pmatrix} 0.00 & 0.34 \\ 1.00 & 0.00 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} -0.34 & 0.00 \\ 0.00 & -1.00 \end{pmatrix}.$$

- Fig.3.1 b

$$N = \begin{pmatrix} 0.30 & 0.34 \\ 0.36 & 0.00 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} -0.64 & 0.00 \\ 0.00 & -0.36 \end{pmatrix}.$$

The results show that for this system the incentive to make an assignment to the second server is greater in the state $x = (1, 0, *, \alpha)$ than in state $x = (1, 0, *, \beta)$ and to the third server is greater in state $x = (1, 1, 0, \alpha)$ than in state $x = (1, 1, 0, \beta)$ if $\nu_\alpha \geq \nu_\beta$.

5.3 $M/E/K$ system

The following Table gives the optimal policies for the system with $K = 5$ Erlangian servers with $m_j = 5$, $j = \overline{1, K}$. To compare the results with the previous models, the mean service intensities $\overline{\mu}_i$ have been chosen to be equal to the corresponding service intensities for the model with exponential servers.

Table 3: Optimal control in each system state

System State x	Number of jobs in the orbit $q(x)$													
$(d_1, d_2, d_3, d_4, d_5)$	0	1	2	3	4	5	6	7	8	9	10	11	12	...
(0,*,*,*,*)	**1**	1	1	1	1	1	1	1	1	1	1	1	1	1
(1,0,*,*,*)	0	0	**2**	2	2	2	2	2	2	2	2	2	2	2
(1,1,0,*,*)	0	0	**3**	3	3	3	3	3	3	3	3	3	3	3
(2,4,0,*,*)	0	0	0	**3**	3	3	3	3	3	3	3	3	3	3
(5,5,0,*,*)	0	0	0	0	**3**	3	3	3	3	3	3	3	3	3
(1,1,1,0,*)	0	0	0	0	**4**	4	4	4	4	4	4	4	4	4
(4,3,1,0,*)	0	0	0	0	0	**4**	4	4	4	4	4	4	4	4
(5,5,4,0,*)	0	0	0	0	0	0	**4**	4	4	4	4	4	4	4
(1,1,1,1,0)	0	0	0	0	0	0	0	**5**	5	5	5	5	5	5
(3,2,1,1,0)	0	0	0	0	0	0	0	0	**5**	5	5	5	5	5
(4,5,2,1,0)	0	0	0	0	0	0	0	0	0	**5**	5	5	5	5

Because of the large state space only selected states are shown in this Table. The fastest server is always activated, whereas all other slower servers have a number of thresholds, which depend on the current service states of faster busy servers. Thus for some fixed phases $(d_1, d_2, \ldots, d_{K+1})$ the threshold sequence looks like $0 = q_1^* \leq q_2^*(d_1) \leq q_3^*(d_1, d_2) \leq \cdots \leq q_K^*(d_1, d_2, \ldots, d_{K+1})$.

As in previous examples the influence of the retrial intensity on the threshold behavior for this model is illustrated in Appendix 2, Figure 2.2:

- γ=0.9 (pictures labeled by letter "b").
- γ=0.3 (pictures labeled by letter "a"),

These diagrams show that the optimal policy has also threshold structure with an analogous behavior as in the previous systems when the intensity γ changes. The thresholds depend on service intensities $\mu_k^{d_k}$ which are different for different phases d_k of the service time distribution.

Some more results for this system with $\lambda = 0.01$ are summarized in the diagrams shown in Figure 3.2 of Appendix 3. The number of phases m_k, $k = \overline{1, K}$ is varied:

- m_k=5 (picture labeled by letter "a"),
- m_k=10 (picture labeled by letter "b").

The stepped curves in these diagrams show that when the residual service time of the faster server decreases, then the incentive to make an assignment to the second server is greater in state $x = (\alpha_1, 0, *)$ than in state $x = (\beta_1, 0, *)$ and to the third server is greater in state $x = (\alpha_1, \alpha_2, 0)$ than in state $x = (\beta_1, \beta_2, 0)$ if $\alpha_k \leq \beta_k$, $k = \{1, 2\}$. We can see that the lower and upper bounds always correspond to the states with the largest residual service time and the smallest residual service time, respectively. The curves for all other possible residual service times lie between these two bounds.

5.4 $MAP/E/K/B + K$ system

The thresholds for this system represent the combined results of the previous systems, it is investigated with the same parameters for the MAP and the Erlang ST-distributions. As in the previous examples the results which are shown in Figure 2.3 of Appendix 2 show the influence of the retrial intensity on the threshold behavior for this model:

- γ=0.9 (pictures labeled by letter "b").

- γ=0.3 (pictures labeled by letter "a"),

The results which are shown in Figure 3.3 of Appendix 3 represent the influence of MAP and ST-phases on the threshold functions $q_2^*(d_1, d_4)$ for the second server. The matrices for the MAP are varied in the same way as for the system $MAP/M/K$ (pictures labeled by letters "a" and "b").

For this type of system the threshold sequence has the form

$$0 = q_1^* \leq q_2^*(d_1, d_{K+1}) \leq q_3^*(d_1, d_2, d_{K+1}) \leq \cdots \leq q_K^*(d_1, \ldots, d_{K-1}, d_{K+1})$$

The incentive to make an assignment to the second server is greater in state $x = (\alpha_1, 0, *, \alpha_4)$ than in state $x = (\beta_1, \beta_2, 0, \beta_4)$ if $\alpha_k \leq \beta_k,\ k = 1, 2$ and $\nu_{\alpha_4} \geq \nu_{\beta_4}$, that is in case of greater residual service time and arrival intensity.

6 Conclusions

In this paper retrial queues with MAP arrivals and phase–type service time distributions have been investigated. It has been shown that the optimal control policy for this class of queueing systems is of threshold type and the threshold function depends on the phases of the arrival and service processes. It has the same monotonicity properties as for the corresponding ordinary queueing systems.

It should be noted that it is very difficult to obtain explicit formulas for the threshold levels. Nevertheless, the numerical analysis permits to investigate the behaviour of optimal control policies when the values of the system parameters are varied. We have presented a novel use of Howard's iteration procedure for retrial queueing systems with heterogeneous servers. This procedure allows us to obtain numerical results and analyze the qualitative properties of optimal control policies for the systems under consideration.

The numerical results show that the threshold level for using the server in the NJM-problem has a weak dependence on the condition of the slower servers, i.e. threshold levels depend mainly on the states of faster servers. Moreover, threshold curves for different states of arrival and service processes lie along the threshold curve for the simple $M/M/K$ retrial queue with mean inter-arrival and service time characteristics of the systems with phases. Therefore, we suspect that in practice the threshold levels for the simple Markovian queue can be quite a good approximation for optimal thresholds for the queues with phases.

The investigation of optimal policy structures and threshold properties can facilitate an implementation of these policies. For each arriving job, a decision maker only has to maintain the information about the number of jobs in the orbit and the collection of busy servers in order to use the optimal server. The decision to switch on some other server can be made via a control table lookup.

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7 Appendices

7.1 Appendix 1. The threshold functions for $M/M/K$ queue.

Fig.1.1. (a) (b)

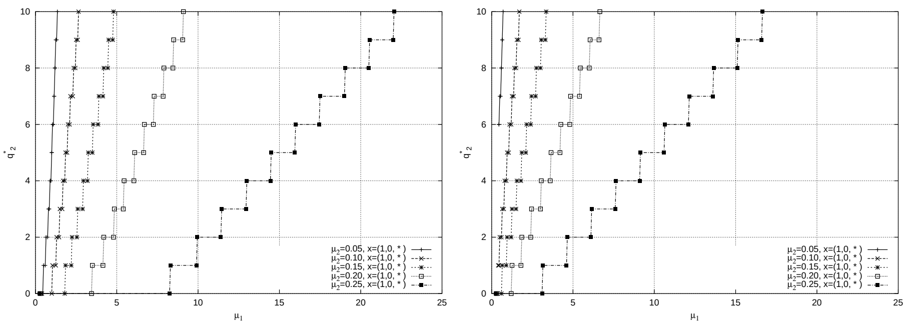


Fig.1.2. (a) (b)

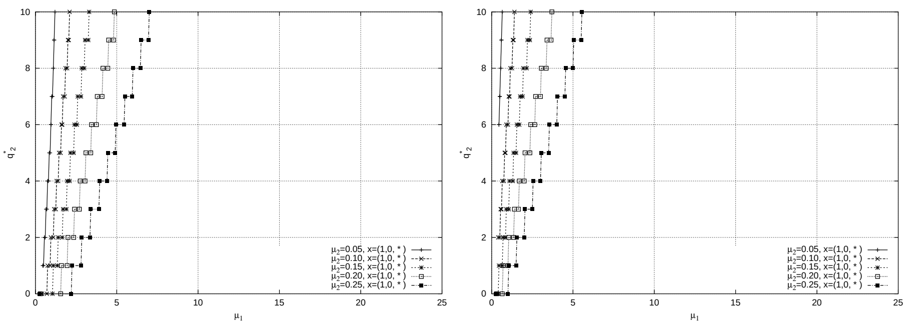
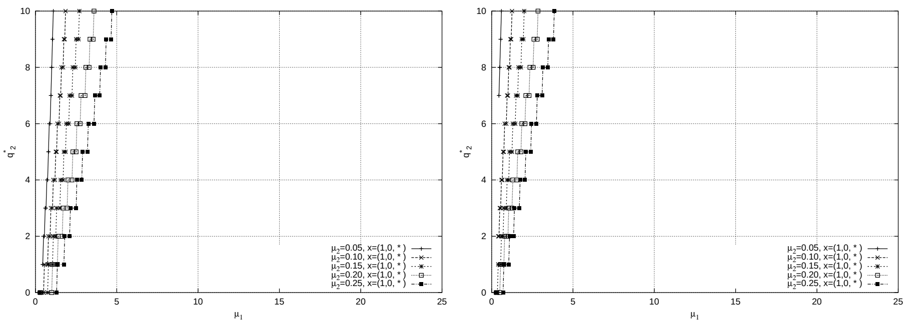


Fig.1.3. (a) (b)



7.2 Appendix 2. The influence of retrial intensity on thresholds functions for different queues.

Fig.2.1. (a) (b)

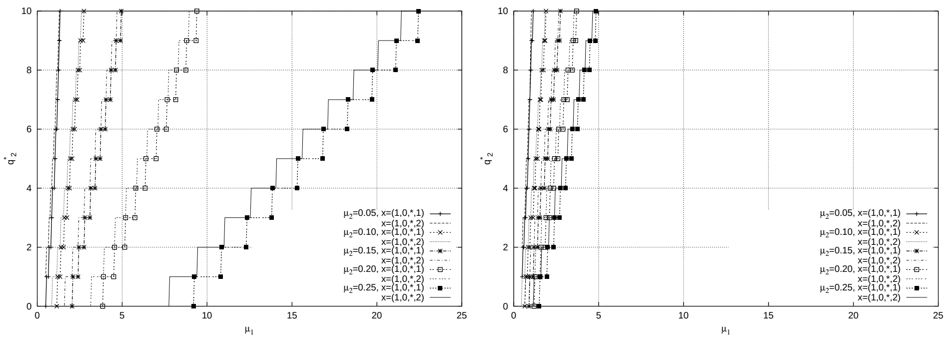


Fig.2.2. (a) (b)

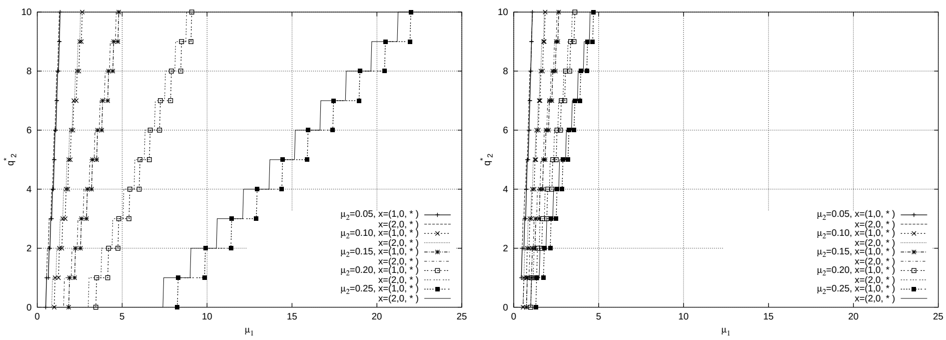
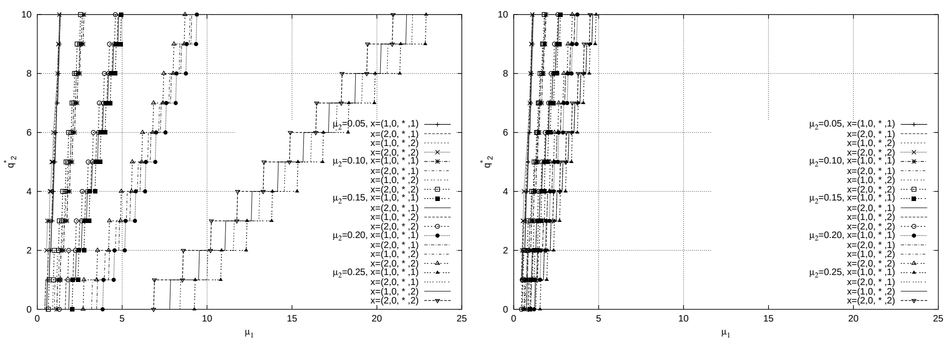
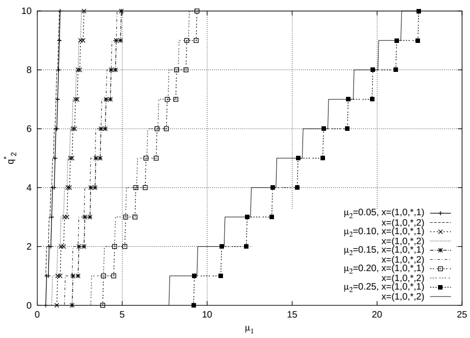


Fig.2.3. (a) (b)



7.3 Appendix 3. The influence of the MAP and Erlang ST-distribution on the threshold functions.

Fig.3.1. (a) (b)



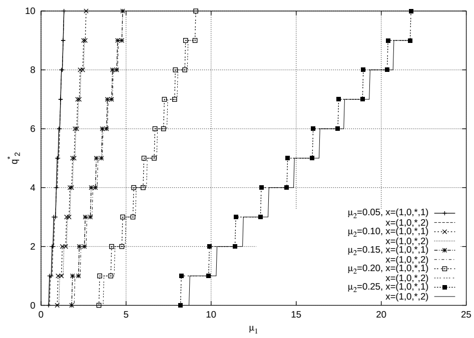
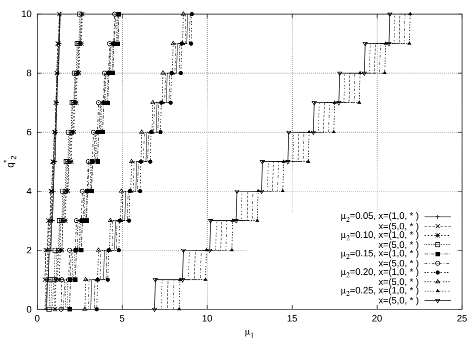


Fig.3.2. (a) (b)



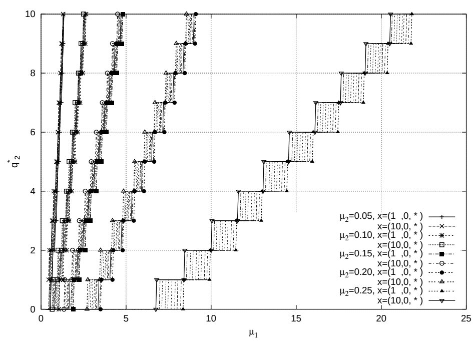


Fig.3.3. (a) (b)