UNIVERSITY OF DEBRECEN
FACULTY OF INFORMATICS

Teaching Queueing Theory and its Applications

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Outline

- Origin of Queueing Theory
- Classifications of Queueing Systems
- Applications
- Solution Methods
- Basic Formulas and Laws
- Reputed Scientists
- Software Support
- References

Origin of Queueing Theory

Agner Krarup Erlang, 1878-1929

- "The Theory of Probabilities and Telephone Conversations", Nyt Tidsskrift for Matematik B, vol 20, 1909.
- "Solution of some Problems in the Theory of Probabilities of Significance in Automatic Telephone Exchanges", Elektrotkeknikeren, vol 13, 1917.
- "The life and works of A.K. Erlang", E. Brockmeyer, H.L. Halstrom and Arns Jensen, Copenhagen: The Copenhagen Telephone Company, 1948.

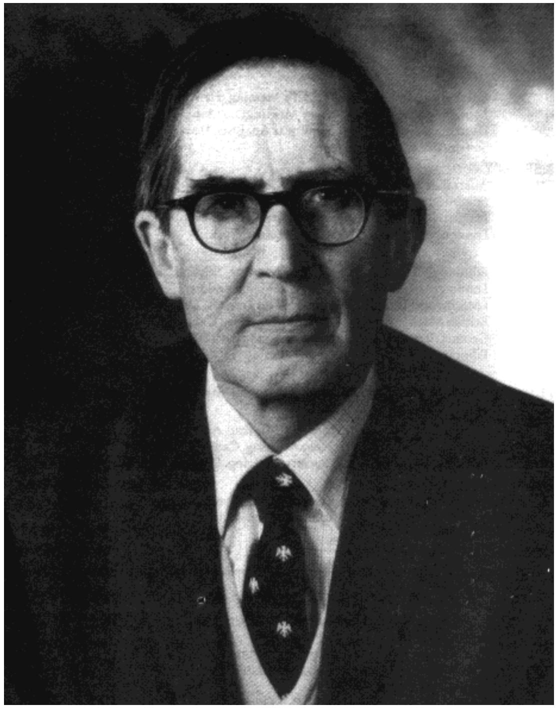
Queueing Theory Homepage

http://web2.uwindsor.ca/math/hlynka/queue.html

Applications

- Telephony, Call Centers
- Manufacturing
- Inventories
- Dams
- Supermarkets
- Computer and Communication Systems
- Sensor Networks, IoT
- Infocommunication Networks, Clouds
- Hospitals
- Many others

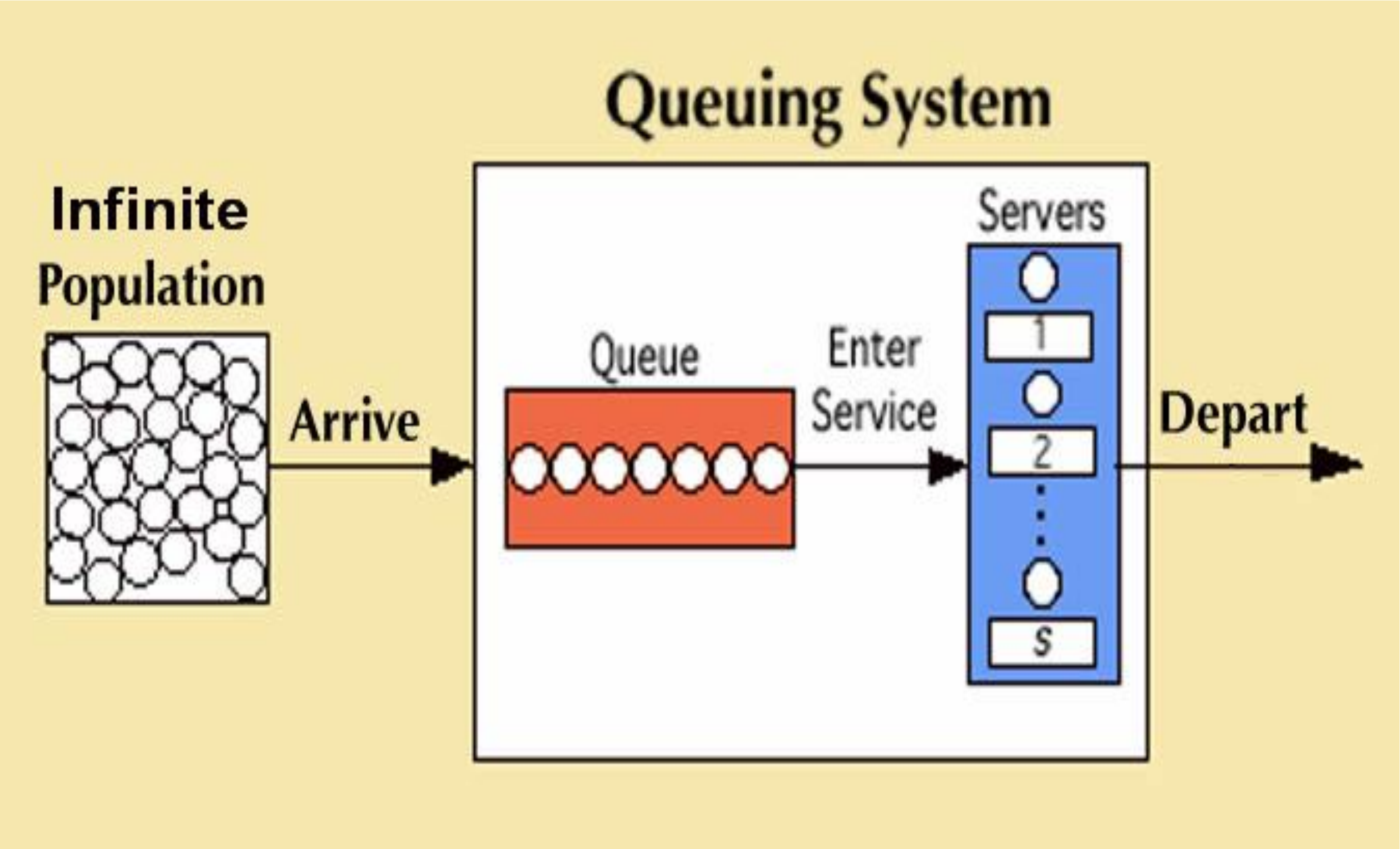
Kendall's Notation



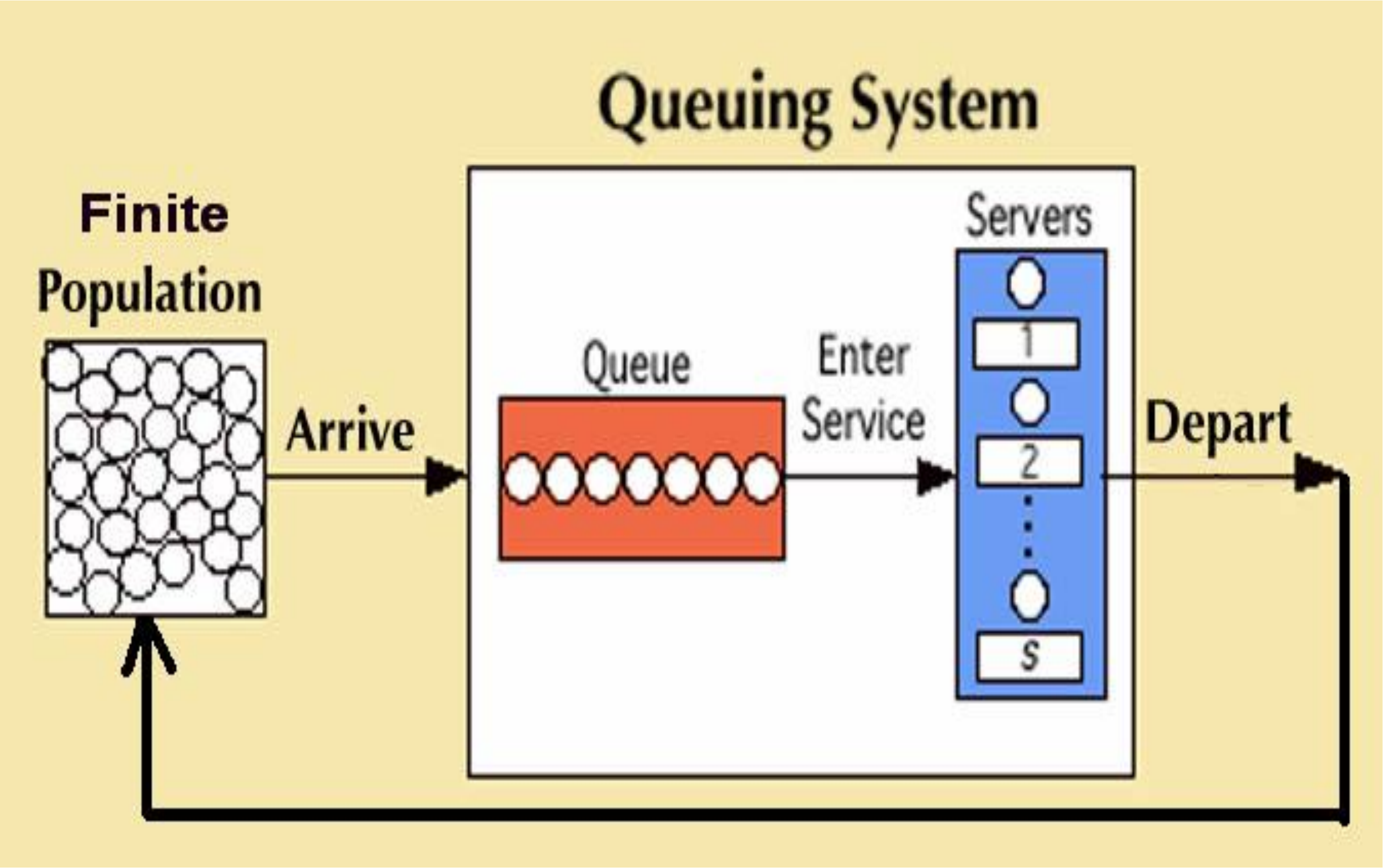
David G. Kendall, 1918-2007

A/B/c/K/m/Z

Infinite-Source Systems



Finite-Source Systems



Performance Metrics

- Utilizations
- Mean Number of Customers in the System / Queue
- Mean Response / Waiting Time
- Mean Busy Period Length of the Server
- Distribution of Response / Waiting Time
- Distribution of the Busy Period
- Distribution of Number of Customers Served during a Busy Period
- Distribution of Number of Retrials until Service Completion

Solution Methodologies

- Analytical
- Numerical
- Asymptotic
- Simulation
- Tool Supported Solutions

Erlang Loss Formula, M/G/c/c Systems

$$B(c,\rho) = p_c = \frac{\rho^c / c!}{\sum_{n=0}^{c} \rho^n / n!} \qquad \rho = \lambda E(B)$$

$$B(c,\rho) = \frac{\rho B(c-1,\rho)/c}{1+\rho B(c-1,\rho)/c} = \frac{\rho B(c-1,\rho)}{c+\rho B(c-1,\rho)}$$

$$B(0,\rho) = 1$$

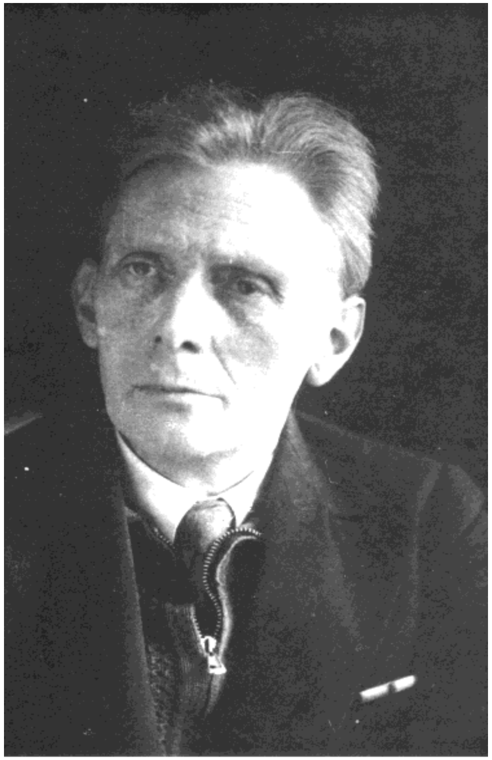
Approximation Formula

$$P_n \approx \frac{\Phi(\frac{n+\frac{1}{2}-\varrho}{\sqrt{\varrho}}) - \Phi(\frac{n-1+\frac{1}{2}-\varrho}{\sqrt{\varrho}})}{\Phi(\frac{n+\frac{1}{2}-\varrho}{\sqrt{\varrho}})} = 1 - \frac{\Phi(\frac{n-\frac{1}{2}-\varrho}{\sqrt{\varrho}})}{\Phi(\frac{n+\frac{1}{2}-\varrho}{\sqrt{\varrho}})},$$

$$\Phi(s) = \int_{-\infty}^{s} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx.$$

Pollaczek-Khintchine Formulas, M/G/1 Systems

Felix Pollachek, 1892-1981



Alexander Y. Khintchine, 1894-1959

Mean Value Formulas

$$C_X^2 := \frac{\mathrm{Var}[X]}{(E[X])^2}$$

$$E[W] = E[R]\frac{\rho}{1-\rho} = \frac{E[B]}{2}\frac{\rho}{1-\rho}(1 + C_B^2)$$

$$E[T] = E[B]\left(1 + \frac{\rho(1 + C_B^2)}{2(1-\rho)}\right)$$

Transform Formulas

$$G_N(z) = L_B(\lambda(1-z)) \cdot \frac{(1-\rho)(1-z)}{L_B(\lambda(1-z)) - z}$$

$$L_T(s) = L_B(s)\frac{s(1-\rho)}{s - \lambda + \lambda L_B(s)}$$

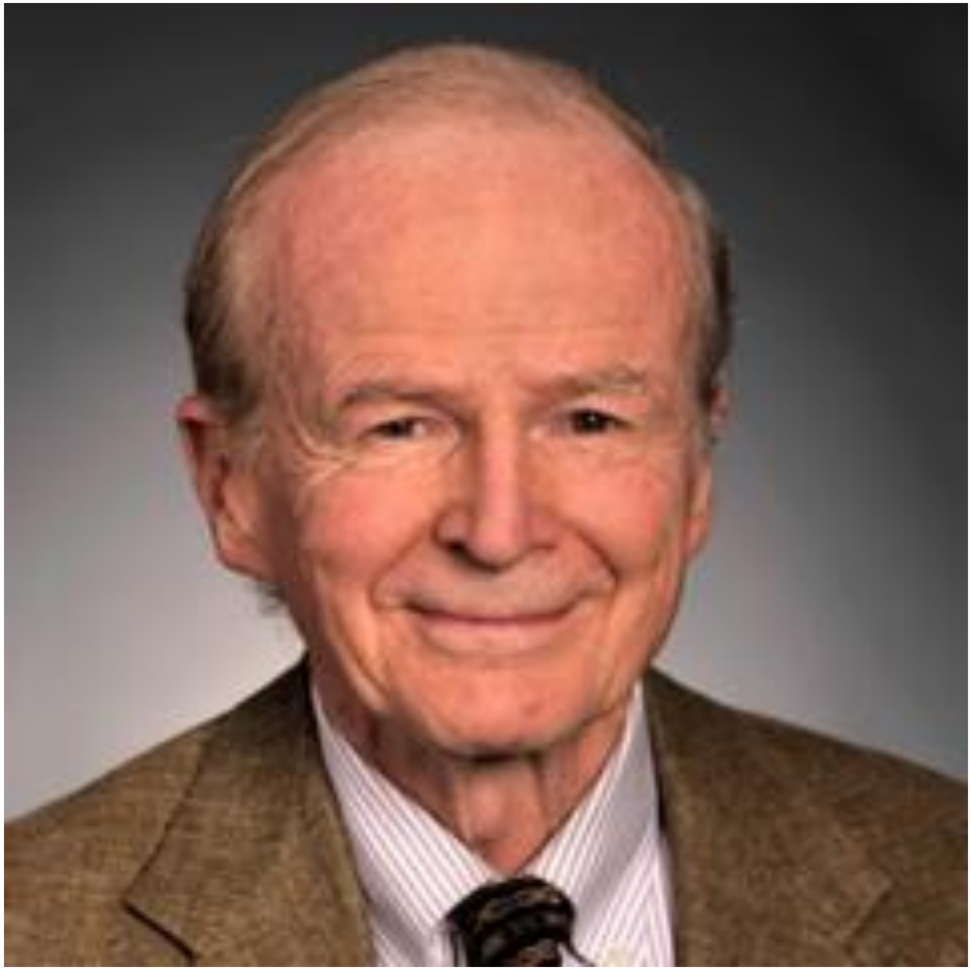
Little's Law

$$\mathbb{E}(N) = \bar{\lambda}\mathbb{E}(T)$$

$$\mathbb{E}(Q) = \bar{\lambda}\mathbb{E}(W)$$

$$\mathbb{E}(N(N-1)\dots(N(-k+1))) = \lambda^k\mathbb{E}(T^k)$$

John Little



John Little, 1928 -

Boris Vladimirovich Gnedenko

Boris Vladimirovich Gnedenko, 1912-1995

Igor Nikolaevich Kovalenko

Igor Nikolaevich Kovalenko, 1935 - 2019

Leonard Kleinrock

Leonard Kleinrock, 1934 -

Lajos Takács

Lajos Takács, 1924 -2015

Takács Formulas, M/G/1 Sytems

$$\mathbb{E}(W^k) = \frac{\lambda}{1-\rho} \sum_{i=1}^{k} \binom{k}{i} \frac{\mathbb{E}(S^{i+1})}{i+1} \mathbb{E}(W^{k-i})$$

$$\mathbb{E}(T^k) = \sum_{l=0}^{k} \binom{n}{l} \mathbb{E}(W^l) \cdot \mathbb{E}(S^{k-l})$$

$$L_\delta(t) = L_S(t + \lambda - \lambda L_\delta(t))$$

Tool Supported Modeling

- University of Dortmund: *HIT, HiQPN, APNN*
  http://ls4-www.infromatik.uni-dortmund.de/tools.html/

- University of Illinois at Urbana-Champaign: *MÖBIUS*
  http://www.mobius.uiuc.edu/

- University of Erlangen: *PEPSY, MOSEL*
  http://www4.informatik.uni-erlangen.de/Projects/MOSEL/

- University of Oxford: *PRISM*
  http://www.prismmodelchecker.org/

Software and Information

http://web2.uwindsor.ca/math/hlynka/qsoft.html

http://mason.gmu.edu/~jshortle/QtsPlus-4-0.zip

QSA (Queueing Systems Assistance)

https://qsa.inf.unideb.hu

Lecture Notes

https://irh.inf.unideb.hu/~jsztrik/education/16/SOR_Main_Angol.pdf

https://irh.inf.unideb.hu/~jsztrik/education/16/Queueing_Problems_Solutions_2021_Sztrik.pdf

Example 1

Customers arrive to a 2 server system according to a Poisson process with rate 3. The service times are exponentially distributed with parameter 2.

Find the minimum capacity of the system for which the probability of blocking is less than 0.01 and the probability that the waiting time exceeds 1.8 minutes is less than 0.05.

Case Studies

Example 2

We have a finite-source system with 50 sources, the request generation times are exponentially distributed with rate 0.5. The service times are exponentially distributed for all the 5 servers with intensity 2.

Find the minimum capacity of the system for which the probability of blocking is less than 0.01 and the probability that the waiting time exceeds 3.5 minutes is less than 0.05.

Case Studies

Example 3

Let us see an M/M/1 system with arrival intensity 1 and the following costs, cost of service per server per unit time CS = 2, cost of waiting in the system per customer per unit time CW =2, cost of idleness per server per unit time CI=10, cost of service rate per server per unit time CSR=10, reward per customer per unit time R=5.

Find the optimal value for the service intensity which minimize the expected total cost per unit time with linear objective function.

Bibliography on Queueing

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- Gnedenko, B.V. – Kovalenko I.N. *Introduction to Queueing Theory, Second Edition*, Birkhauser, 1989
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- Kleinrock, L. *Queueing Systems, Vol. I-II*, John Wiley Sons, 1976
- Takács, L. *Introduction to the Theory of Queues*, Oxfrod University Press, 1962

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- Allen, A.O. *Probability, Statistics, and Queueing Theory with Computer Science Applications, 2nd Edition*, Academic Press, 1990
- Dattatreya, G. *Performance Analysis of Queueing and Computer Networks*, CRC Press, 2008
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- Cai, L., Shen, X., Mark, J.W. *Multimedia Services in Wireless Internet, Modeling and Analysis*, John Wiley Sons, 2009
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- Kouvatsos, D. *Network Performance Engineering, A Handbook on Convergent Multi-Service Networks and Next Generation Internet*, Springer, 2011
- Misis, J., Misic, V.B. *Performance Modeling and Analysis of Bluetooth Networks: Polling, Scheduling and Traffic Control*, Auerbach Publications, 2006

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