Queueing Theory with Applications
A Personal View

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Outline

- Origin of Queueing Theory
- Classifications of Queueing Systems
- Applications
- Solution Methods
- Basic Formulas and Laws
- Hungarian Contributions
- Recent Developments
- References
Origin of Queueing Theory

Agner Krarup Erlang, 1878-1929

Queueing Theory Homepage
http://web2.uwindsor.ca/math/hlynka/queue.html
Murphy’s Law of Queue

- If you change queues, the one you have left will start to move faster than the one you are in now.
- Your queue always goes the slowest
- Whatever queue you join, no matter how short it looks, will always take the longest for you to get served.

Google search for “Queueing Theory “: 188 000
Applications

- Telephony
- Manufacturing
- Inventories
- Dams
- Supermarkets
- Computer and Communication Systems
- Call Centers
- Infocommunication Networks
- Hospitals
- Many others
Kendall’s Notation

David G. Kendall, 1918-2007

A/B/c/K/m/Z
Performance Metrics

- Utilizations
- Mean Number of Customers in the System / Queue
- Mean Response / Waiting Time
- Mean Busy Period Length of the Server
- Distribution of Response / Waiting Time
- Distribution of the Busy Period
Solution Methodologies

- Analytical
- Numerical
- Asymptotic
- Simulation
- Tools
Erlang Loss Formula, M/G/c/c Systems

\[ B(c, \rho) = p_c = \frac{\rho^c / c!}{\sum_{n=0}^{c} \rho^n / n!} \]

\[ \rho = \lambda E(B) \]

\[ B(c, \rho) = \frac{\rho B(c - 1, \rho)/c}{1 + \rho B(c - 1, \rho)/c} = \frac{\rho B(c - 1, \rho)}{c + \rho B(c - 1, \rho)} \]

\[ B(0, \rho) = 1 \]
Approximation Formula

\[ B(n, \rho) \approx \frac{\Phi(s) - \Phi(s - 1)}{\Phi(s)} = 1 - \frac{\Phi(s - 1)}{\Phi(s)}, \]

\[ \Phi(s) = \int_{-\infty}^{s} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx, \quad s = \frac{n + \frac{1}{2} - \zeta}{\sqrt{\zeta}}. \]

http://www.erlang.com/calculator/
http://jani.uw.hu/erlang/erlang.html
Erlang Delay Formula
M/M/n Systems

\[ C(n, \rho) = \frac{nB(n, \rho)}{n - \rho + \rho B(n, \rho)}, \]

\[ C(n, \rho) = \frac{\rho(n - 1 - \rho) \cdot C(n - 1, \rho)}{(n - 1)(n - \rho) - \rho C(n - 1, \rho)}, \quad C(1, \rho) = \rho \]
Square-root Approximation Formula

\[ C(n^*_\alpha, \rho) < \alpha, \quad n^*_\alpha = \rho + k_\alpha \sqrt{\rho} \]

\[ k_\alpha \frac{\phi(k_\alpha)}{\varphi(k_\alpha)} = \frac{1 - \alpha}{\alpha} \]

\[ \alpha = 0.8, 0.5, 0.2, 0.1, \quad k_\alpha = 0.1728, 0.5061, 1.062, 1.420 \]
Pollaczek-Khintchine Formulas, M/G/1 Systems

Felix Pollachek, 1892-1981

Alexander Y. Khintchine, 1894-1959
Mean Value Formulas

\[ C_X^2 := \frac{\text{Var}[X]}{(E[X])^2} \]

\[ E[W] = E[R] \frac{\rho}{1 - \rho} = \frac{E[B]}{2} \frac{\rho}{1 - \rho} (1 + C_B^2) \]

\[ E[T] = E[B] \left( 1 + \frac{\rho(1 + C_B^2)}{2(1 - \rho)} \right) \]
Transform Formulas

\[ G_N(z) = L_B(\lambda(1 - z)) \cdot \frac{(1 - \rho)(1 - z)}{L_B(\lambda(1 - z)) - z} \]

\[ L_T(s) = L_B(s) \frac{s(1 - \rho)}{s - \lambda + \lambda L_B(s)} \]
Little’s Law

\[ \mathbb{E}(N) = \bar{\lambda}\mathbb{E}(T) \]

\[ \mathbb{E}(Q) = \bar{\lambda}\mathbb{E}(W) \]

\[ \mathbb{E}(N(N-1) \ldots (N(-k+1))) = \lambda^k \mathbb{E}(T^k) \]
Reputed Scientists

Boris Vladimirovich Gnedenko, 1912-1995
Reputed Scientists

Leonard Kleinrock, 1934 -
Hungarian Contributions

Lajos Takács, 1924 -
Takács Formulas, M/G/1 Systems

\[ \mathbb{E}(W^k) = \frac{\lambda}{1 - \rho} \sum_{i=1}^{k} \binom{k}{i} \frac{\mathbb{E}(S^{i+1})}{i + 1} \mathbb{E}(W^{k-i}) \]

\[ \mathbb{E}(T^k) = \sum_{l=0}^{k} \binom{n}{l} \mathbb{E}(W^l) \cdot \mathbb{E}(S^{k-l}) \]

\[ L_\delta(t) = L_S(t + \lambda - \lambda L_\delta(t)) \]
Hungarian Contributions

- Eötvös Loránd University
  (A. Benczúr, L. Lakatos, L. Szeidl)

- Budapest University of Technology and Economics
  (L. Györfi, M. Telek, S. Molnár)

- University of Debrecen
  (J. Tomkó, M. Arató, B. Almási, A. Kuki, J. Sztrik)
Java Applets and Information


http://irh.inf.unideb.hu/user/jsztrik/
Recent Developments

23nd International Teletraffic Congress

September 6-8, 2011, San Francisco, USA

• Network architectures, paradigms and technologies
• Network planning, QoS, and associated performance issues
• Traffic management and measurement
• Applications
• Security issues
• Models and techniques
QTNA, 2012, Kyoto, Japan
Tool supported modeling

- University of Dortmund: *HIT, HiQPN, APNN*
  http://ls4-www.infromatik.uni-dortmund.de/tools.html/

- University of Illinois at Urbana-Champaign: *MÖBIUS*
  http://www.mobius.uiuc.edu/

- University of Erlangen: *PEPSY, MOSEL*
  http://www4.informatik.uni-erlangen.de/Projects/MOSEL/

- University of Oxford: *PRISM*
  http://www.prismmodelchecker.org/
Softwares and Information

http://web2.uwindsor.ca/math/hlynka/qsoft.html
Bibliography on Queueing


Bibliography on Applications


Thank You for Your Attention