Asymptotic methods in performance modelling of finite-source retrial queues with collisions and their applications in smart city networks

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Outline

1. Finite source retrial queueing system with collisions
2. Performance measures
3. Tool supported, algorithmic, simulation and asymptotic approaches
4. Comparisons, examples
5. Bibliography
Finite source retrial queueing system with collisions

Figure 1: Finite source retrial queueing system with collisions and unreliable server

$N$
Performance measures

- Distribution of number of requests in the system, including in service and in orbit

- Distribution of number of retrials

- Distribution of the response/waiting time of a customer
Tool supported and algorithmic approaches

- **MOSEL (Modeling, Specification and Evaluation Language) solution**

- **Algorithmic method**
Simulation approach

- The effect of distributions of the involved random variables on the distribution of the number of customers in the system

- The effect of distributions of the involved random variables on the mean and variance of the response/waiting time of a request

- The effect of distributions of the involved random variables on the mean and variance of the number of retrials
Asymptotic method

\[ B(x) = \gamma_0 \gamma_1 \gamma_2 \]
Asymptotic of the first order

Let \( Q(\infty) \) be the number of customers in the system in steady-state, then

\[
\lim_{N \to \infty} E \exp \left\{ i\omega \frac{Q(\infty)}{N} \right\} = \exp \{ i\omega \kappa_1 \},
\]

where the value of parameter \( \kappa_1 \) is the positive solution of the equation

\[
\lambda (1 - \kappa_1) - a[\kappa_1] [R_0[\kappa_1] - R_1[\kappa_1]] + \gamma_1 R_1[\kappa_1] = 0,
\]
here $a [\kappa_1]$ is
\[
a [\kappa_1] = \lambda (1 - \kappa_1) + \sigma \kappa_1, \tag{3}
\]
and the stationary distributions of probabilities $R_k[\kappa_1]$ of the service state $k$ are defined as follows
\[
R_0[\kappa_1] = \left\{ \frac{\gamma_0 + \gamma_2}{\gamma_2} + \frac{\gamma_1 + \gamma_2}{\gamma_2} \cdot \frac{a [\kappa_1]}{a [\kappa_1] + \gamma_1} \left[ 1 - B^* (a [\kappa_1] + \gamma_1) \right] \right\}^{-1},
\]
\[
R_1[\kappa_1] = R_0[\kappa_1] \frac{a [\kappa_1]}{a [\kappa_1] + \gamma_1} \cdot \left[ 1 - B^* (a [\kappa_1] + \gamma_1) \right],
\]
\[
R_2[\kappa_1] = \frac{1}{\gamma_2} \left[ \gamma_0 R_0[\kappa_1] + \gamma_1 R_1[\kappa_1] \right]. \tag{4}
\]
Asymptotic of the second order

\[
\lim_{N \to \infty} E \exp \left\{ iw \frac{Q(\infty) - \kappa_1 N}{\sqrt{N}} \right\} = \exp \left\{ \frac{(iw)^2}{2} \kappa_2 \right\},
\]

(5)

the value of parameter \( \kappa_2 \) is defined by expression

\[
\kappa_2 = \frac{\lambda (1 - \kappa_1) \left\{ b_2 [1 - b_3] + \lambda (1 - \kappa_1) (a + \gamma_1) R_2 + b_1 [R_0 - b_3] \right\}}{\lambda b_2 + (\sigma - \lambda) \left\{ (b_1 + b_2) [b_3 - R^*_1(a + \gamma_1)] - b_1 [R_0 - R_1] \right\}},
\]

(6)

where

\[
b_1 = \lambda (1 - \kappa_1) (\gamma_1 + \gamma_2), \quad b_2 = \gamma_2 [a + \gamma_1], \quad b_3 = R_0 B^*(a + \gamma_1).
\]
From the proved theorem it follows that if \( N \to \infty \) the limiting distributions for the centered and normalized number of customers in the system has a Gaussian distribution with variance \( \kappa_2 \), defined by the expression (6).

**Corollary**

As a consequence the distribution of the number of customers in the system is Gaussian with mean \( N\kappa_1 \) and variance \( N\kappa_2 \), respectively.
Generating function of the number of retrials

For the generating function of the number of transitions $\nu$ of the tagged customer into the orbit we have

$$\lim_{N \to \infty} E z^{\nu} = \frac{1 - q}{1 - qz},$$

(7)

where the probability $q$ can be obtained as

$$q = 1 - R_0 B^*(a + \gamma_1),$$

(8)
Consequently, $\nu$ is geometric, namely

$$P \{ \nu = n \} = (1 - q)q^n, \quad n = 0, \infty,$$

and for the prelimit situation, that is when $N$ is fixed we can and will use the following approximation $P \{ \nu = n \} \approx (1 - q)q^n$. 
Characteristic function of the waiting time $W$ of the tagged customer in the orbit

\[ \mathbb{E}e^{iuW} \approx (1 - q) + q \frac{\sigma(1 - q)}{\sigma(1 - q) - iuN}. \]  \hspace{1em} (10)
The average sojourn time $\bar{T}_S$ of the customer under service

$$\bar{T}_S \approx \frac{1 - B^*(a + \gamma_1)}{(a + \gamma_1)B^*(a + \gamma_1)}.$$  \hspace{1cm} (11)
Comparisons, examples

For the considered retrial queuing system we choose gamma distributed service time $S$ with a shape parameter $\alpha$ and scale parameter $\beta$, with Laplace-Stieltjes transform $B^*(\delta)$ of the form

$$B^*(\delta) = \left(1 + \frac{\delta}{\beta}\right)^{-\alpha},$$

in the case when $\alpha = \beta$, that is when the average service time is equal to unit.

It can be shown that

$$\mathbb{E}(S) = \frac{\alpha}{\beta}, \quad \text{Var}(S) = \frac{\alpha}{\beta^2}, \quad V_S^2 = \frac{1}{\alpha},$$

where $V_S^2$ denotes the squared coefficient of variation of $S$. 
Input parameters:

\[ \lambda = 1 \, , \, \sigma = 5 \, , \, \gamma_0 = 0.1 \, , \, \gamma_1 = 0.2 \, , \, \gamma_2 = 1 \, , \, \alpha = \beta = 1.778 \]
Comparison of the Gaussian approximation and numerical results in the case $N = 10$
Comparison of the Gaussian approximation and numerical results in the case $N = 30$
Comparison of the Gaussian approximation and numerical results in the case $N = 50$
To determine the accuracy and area of applicability of Gaussian approximation we will use the Kolmogorov-distance which can be defined as follows

\[ \Delta = \max_{0 \leq j \leq N} \left| \sum_{n=0}^{j} (\Pi(n) - P_{as}(n)) \right| . \]
Table 1: Kolmogorov-distance between prelimit distribution $\Pi(j)$ and its normal approximation $P_{as}(j)$ for various values of $N$ and $\alpha = \beta$

<table>
<thead>
<tr>
<th>V</th>
<th>$\alpha$</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>50</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>$\alpha = 4$</td>
<td>0.090</td>
<td>0.068</td>
<td>0.039</td>
<td>0.030</td>
<td>0.024</td>
<td>0.018</td>
</tr>
<tr>
<td>0.75</td>
<td>$\alpha = 1,778$</td>
<td>0.065</td>
<td>0.042</td>
<td>0.029</td>
<td>0.023</td>
<td>0.018</td>
<td>0.013</td>
</tr>
<tr>
<td>1</td>
<td>$\alpha = 1$</td>
<td>0.037</td>
<td>0.023</td>
<td>0.017</td>
<td>0.014</td>
<td>0.011</td>
<td>0.007</td>
</tr>
<tr>
<td>1.5</td>
<td>$\alpha = 0,444$</td>
<td>0.029</td>
<td>0.006</td>
<td>0.004</td>
<td>0.004</td>
<td>0.003</td>
<td>0.002</td>
</tr>
<tr>
<td>3</td>
<td>$\alpha = 0,111$</td>
<td>0.010</td>
<td>0.068</td>
<td>0.030</td>
<td>0.019</td>
<td>0.016</td>
<td>0.014</td>
</tr>
</tbody>
</table>
Running the simulation program with inputs

\[ \lambda = 1, \quad \sigma = 1, \quad \gamma_0 = 0.1, \quad \gamma_1 = 0.1, \quad \gamma_2 = 1, \]

and applying the proposed approximation (9) we calculate the
Kolmogorov distance \( \Delta \) for various values of \( N \) and \( \alpha = \beta \) in
Table 2.

**Table 2:** Kolmogorov distance between distribution of retrials \( P_s(n) \) and \( P_{as}(n) \) for various values of parameters \( N \) and \( \alpha = \beta \)

<table>
<thead>
<tr>
<th></th>
<th>( N = 10 )</th>
<th>( N = 30 )</th>
<th>( N = 50 )</th>
<th>( N = 70 )</th>
<th>( N = 100 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha = 0.5 )</td>
<td>0.0218</td>
<td>0.0067</td>
<td>0.0038</td>
<td>0.0029</td>
<td>0.0021</td>
</tr>
<tr>
<td>( \alpha = 1 )</td>
<td>0.0292</td>
<td>0.0099</td>
<td>0.0064</td>
<td>0.0048</td>
<td>0.0035</td>
</tr>
<tr>
<td>( \alpha = 2 )</td>
<td>0.0360</td>
<td>0.0119</td>
<td>0.0075</td>
<td>0.0056</td>
<td>0.0040</td>
</tr>
</tbody>
</table>
Conclusions

1. Finite source retrial queueing system with collisions
2. Different solution approaches
3. Recent results on systems with an unreliable server
4. Graphical illustrations, comparisons, examples
Tool Supported and Numerical Methods

Numerical analysis of retrial queueing systems with conflict of customers,

An Algorithmic Approach for the Analysis of Finite-Source $M/GI/1$ Retrial Queueing Systems with Collisions and Server Subject to Breakdowns and Repairs,
Simulation Methods

Simulation of Finite-Source Retrial Queueing Systems with Collisions and Non-reliable Server,

Comparison of two operation modes of finite-source retrial queueing systems with collisions and non-reliable server by using simulation,
Asymptotic Methods

A. Nazarov – J. Sztrik – A. Kvach
Some Features of a Finite-Source $M/GI/1$ Retrial Queuing System with Collisions of Customers,

A. Nazarov – J. Sztrik – A. Kvach
A Survey of Recent Results in Finite-Source Retrial Queues with Collisions,
Asymptotic Analysis of Finite-Source $M/GI/1$ Retrial Queueing Systems with Collisions and Server Subject to Breakdowns and Repairs,

A. Nazarov – J. Sztrik – A. Kvach
Asymptotic Waiting Time Analysis of Finite Source $M/GI/1$ Retrial Queueing Systems with Conflicts and Unreliable Server,
Thank You for Your Attention